

# Lecture 6

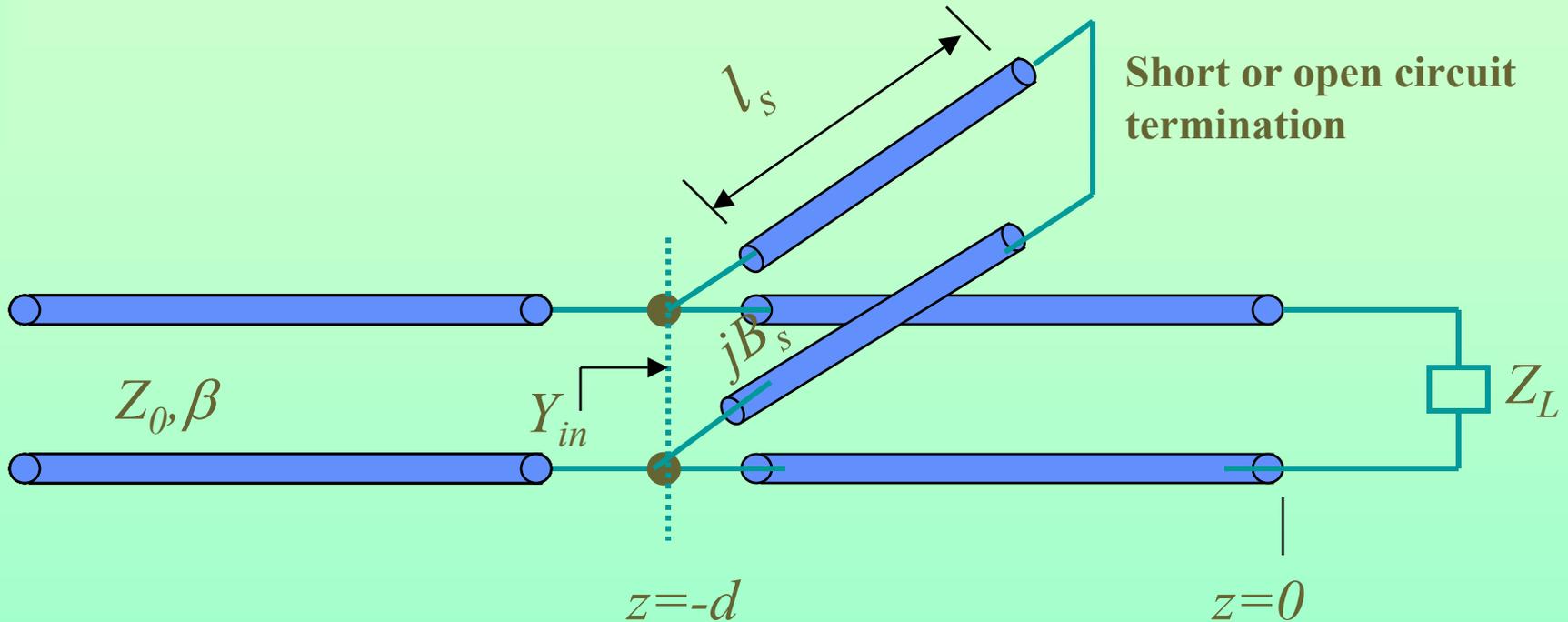
■ Stub – Matching

■ Waveguides

November 5, 2002

# Single Stub Matching

The single-stub tuner uses a shorted or open section of TL attached at some position along another TL:



# Single Stub Matching

Why an open or shorted section? Because these are easy to fabricate, the length can easily be made adjustable and little to no power is dissipated in the stub.

We will study SST from two perspectives. First, we will develop an analytical solution. Second, we will use SC to obtain a graphical solution.

Last week we derived the transformed load impedance for a TL as:

$$Z = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} = Z_0 \frac{Z_L + jZ_0 t}{Z_0 + jZ_L t} \quad (t = \tan \beta d)$$

# Single Stub Matching

With parallel connections, it is much simpler to work with admittances than impedances. We will define the transformed load admittance as  $Y = 1/Z = G + jB$ .

The distance  $d$  is chosen so that  $G = Y_0 (= 1/Z_0)$ . This condition leads to the solutions

$$\frac{d}{\lambda} = \begin{cases} \frac{1}{2\pi} \tan^{-1} t, & t \geq 0 \\ \frac{1}{2\pi} (\pi + \tan^{-1} t), & t \leq 0 \end{cases}$$

# Single Stub Matching

Where

$$t = \begin{cases} X_L \pm \sqrt{\left(\frac{R_L}{Z_0}\right) \left[ (Z_0 - R_L)^2 + X_L^2 \right]} & R_L \neq Z_0 \\ -\frac{X_L}{2Z_0}, & R_L = Z_0 \end{cases}$$

and

$$Z_L = R_L + jX_L$$

# Single Stub Matching

With this location of the stub, the transformed load admittance has a real part  $Y_0$  which is almost a matched state. In general, however, this transformed  $Y_L$  will also have an imaginary part  $B$ .

The length of the stub,  $l_s$  is chosen so that its input susceptance  $B_s = -B$ . Consequently, the parallel combination of the stub input susceptance and the transformed load admittance yield an input admittance  $Y_{in} = Y_0$  as seen from the source end of the TL.

This condition provides the solutions

$$\frac{l_s}{\lambda} = \frac{1}{2\pi} \tan^{-1} \left( \frac{Y_0}{B} \right) \quad (\text{short-circuit stub})$$

$$\frac{l_s}{\lambda} = -\frac{1}{2\pi} \tan^{-1} \left( \frac{B}{Y_0} \right) \quad (\text{open-circuit stub})$$

# Single Stub Matching

Where  $B$  is the transformed load susceptance at  $z=-d$ . Lengths of TL that are integer multiples of  $\lambda/2$  can be added or subtracted without altering the tuning.

## Graphical Solution Using Smith Chart

Using the Smith chart, design a shorted single-stub tuner to match the load  $Z_L=35-j47.5 \ \Omega$  to a TL with characteristic impedance  $Z_0=50 \ \Omega$ .

# SST- Smith Chart

The normalized load impedance and admittance are:

$$z_L = 0.7 - j0.68 \text{ and } y_L = 0.5 + j0.68$$

**STEP 1:** Locate  $y_L = 0.5 + j0.68$  on the Smith admittance chart.

**STEP 2:** Draw the constant VSWR.

**STEP 3:** Draw the line segment from the origin to  $y_L$ . Rotate this vector towards the source until it intersects the unit conductance circle. Along this circle  $\text{Re}[y(z)] = 1$ .

This is really the intersection of constant VSWR circle for this load with unit conductance circle.

# SST- Smith Chart

There will be two solutions. Both of these give  $y_1' = 1 + jb_1$ . For this example we find from SC that:

$$I \quad y' = 1 + j1.2$$

$$II \quad y' = 1 - j1.2$$

**STEP 4:** From these rotations we can determine  $d$  as

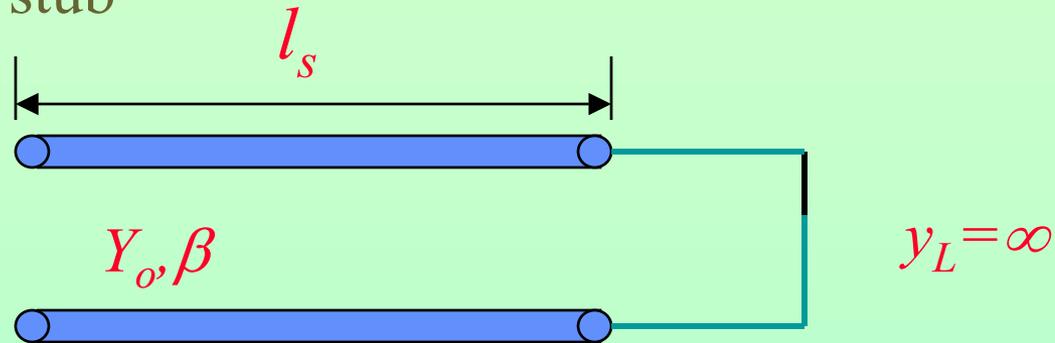
$$I \quad d = 0.168\lambda - 0.109\lambda = 0.059\lambda$$

$$II \quad d = 0.332\lambda - 0.109\lambda = 0.223\lambda$$

When either of these two susceptances is added to  $y_1'$ , then  $y_{in} = 1$ .

# SST- Smith Chart

The stub lengths can be determined directly from the Smith chart.  
Consider the short stub



On the Smith admittance chart,  $y_L = \infty$  is located at  $\text{Re}\{\Gamma\}=1$ ,  
 $\text{Im}\{\Gamma\}=0$ . From there, rotate “wavelengths towards generator” to:

$$I \quad b_s = -1.2 \Rightarrow \ell_s = 0.361\lambda - 0.25\lambda = 0.111\lambda$$

$$II \quad b_s = +1.2 \Rightarrow \ell_s = 0.25\lambda + 0.139\lambda = 0.389\lambda$$

# SST- Smith Chart

∴ The final two solutions are:

$$I \quad d = 0.059\lambda \quad \text{and} \quad \ell_s = 0.111\lambda$$

$$II \quad d = 0.223\lambda \quad \text{and} \quad \ell_s = 0.389\lambda$$

## Disadvantage of single stubs

Single stub placement means that we have no choice in the position (distance from the load) of the stub. This not might be practical. Furthermore, not all load impedances can be matched!

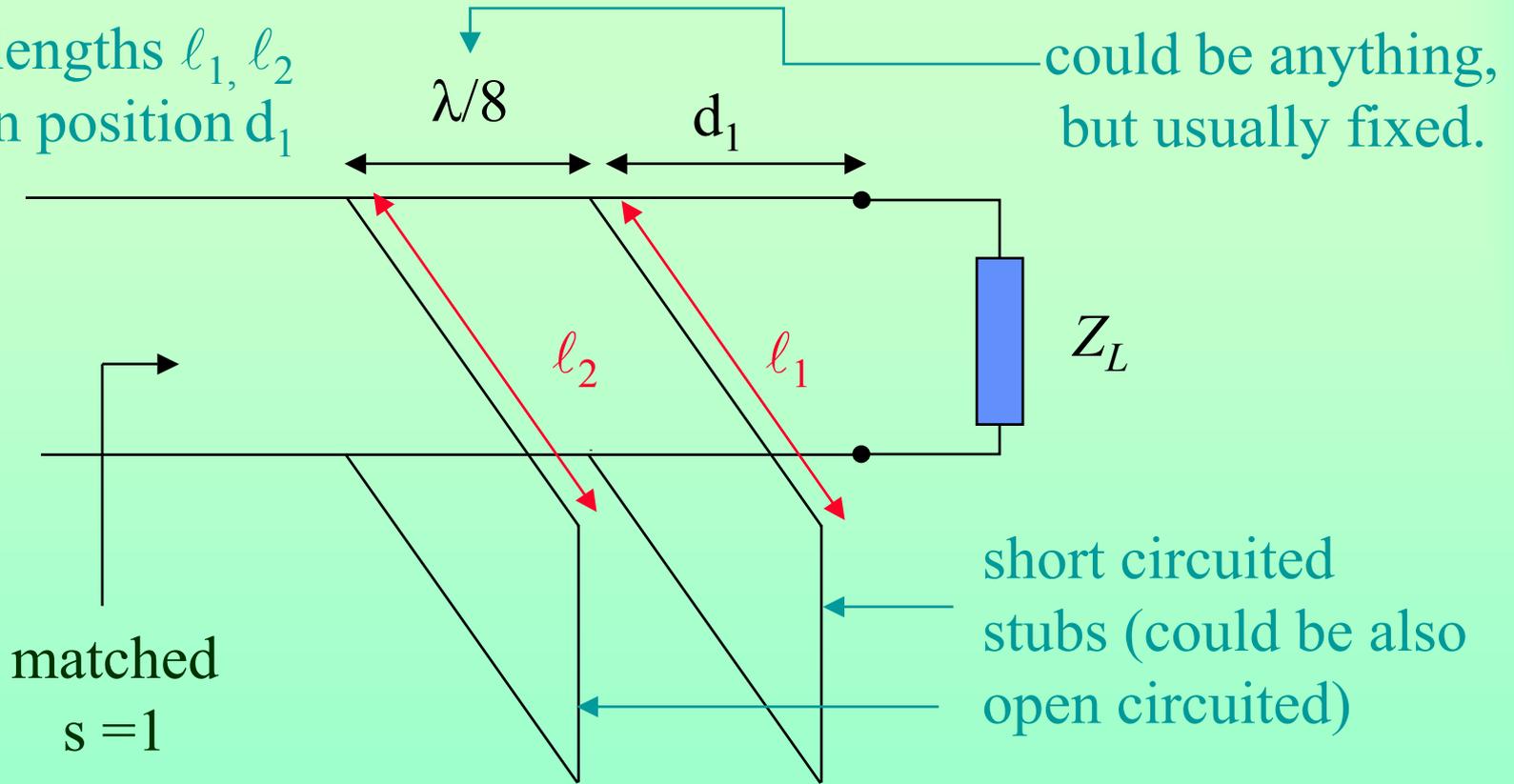
A **double stub tuner** does not have this problem, the distance from the load is more or less arbitrary.

However, it still cannot match **all** impedances.

# Sample Arrangement

determine

stub lengths  $l_1, l_2$   
given position  $d_1$



# Wave guide

# Transverse and Longitudinal Components

break up the operator into transverse & longitudinal components

$$\begin{aligned}\nabla^2 \vec{E} &= (\nabla_{xy}^2 + \nabla_z^2) \vec{E} \\ &= \left( \nabla_{xy}^2 + \frac{\partial^2}{\partial z^2} \right) \vec{E} \\ &= (\nabla_{xy}^2 + \gamma^2) \vec{E}\end{aligned}$$

The wave equations become now

$$\begin{aligned}\nabla_{xy}^2 \vec{E} + (\gamma^2 + k^2) \vec{E} &= 0 \\ \nabla_{xy}^2 \vec{H} + (\gamma^2 + k^2) \vec{H} &= 0\end{aligned}$$

## **Solution strategy**

We still have (seemingly) six simultaneous equations to solve. In fact, the 6 are NOT independent. This looks complicated! Adopt a strategy of expressing the transverse fields (the  $E_x, E_y, H_x, H_y$  components in terms of the longitudinal components  $E_z$  and  $H_z$  only. If we can do this we only need find  $E_z$  and  $H_z$  from the wave equations....Too easy eh!

The first step can be carried out directly from the two curl equations from the original Maxwell's eqns. Writing these out:

## First step

$$\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x \quad (1) \quad \frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x \quad (4)$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \quad (2) \quad -\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (5)$$

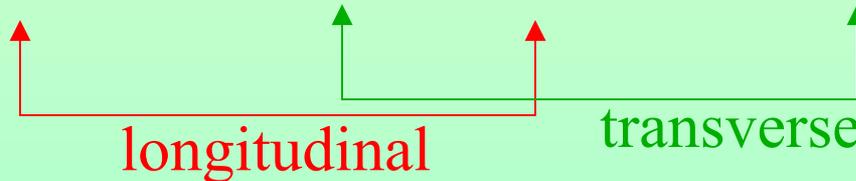
$$\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (3) \quad \frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad (6)$$

All  $\frac{\partial}{\partial z}$  replaced by  $-\gamma$ . All fields are functions of  $x$  and  $y$  only.

# Result

Now, manipulate to express the transverse in terms of the longitudinal. E.g. From (1) and (5) eliminate  $E_y$

$$\frac{\partial E_z}{\partial y} + \frac{\gamma}{j\omega\epsilon} \left( -\gamma H_x - \frac{\partial H_z}{\partial x} \right) = -j\omega\mu H_x$$



$$\Rightarrow H_x = \frac{-1}{k_c^2} \left( \gamma \frac{\partial H_z}{\partial x} - j\omega\epsilon \frac{\partial E_z}{\partial y} \right)$$

where  $k_c^2 = \gamma^2 + k^2$

$k_c$  is an eigenvalue  
(to be discussed)

# The other components

$$H_x = \frac{-1}{k_c^2} \left( \gamma \frac{\partial H_z}{\partial x} - j\omega\epsilon \frac{\partial E_z}{\partial y} \right)$$

$$H_y = \frac{-1}{k_c^2} \left( \gamma \frac{\partial H_z}{\partial y} + j\omega\epsilon \frac{\partial E_z}{\partial x} \right)$$

$$E_x = \frac{-1}{k_c^2} \left( \gamma \frac{\partial E_z}{\partial x} + j\omega\mu \frac{\partial H_z}{\partial y} \right)$$

$$E_y = \frac{-1}{k_c^2} \left( \gamma \frac{\partial E_z}{\partial y} - j\omega\mu \frac{\partial H_z}{\partial x} \right)$$

So find solutions for  $E_z$  and  $H_z$  and then use these 4 eqns to find all the transverse components

We only need to find  $E_z$  and  $H_z$  now!

## Wave type classification

It is convenient to to classify as to whether  $E_z$  or  $H_z$  exists according to:

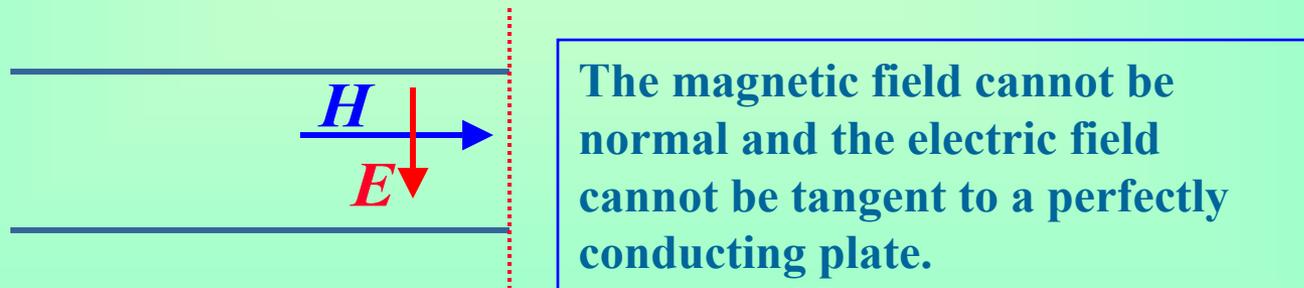
TEM:	$E_z = 0$	$H_z = 0$
TE:	$E_z = 0$	$H_z \neq 0$
TM	$E_z \neq 0$	$H_z = 0$

We will first see how TM wave types propagate in waveguide  
Then we will infer the properties of TE waves.

The TE modes of a parallel plate wave guide are preserved if perfectly conducting walls are added perpendicularly to the electric field.



On the other hand, TM modes of a parallel wave guide disappear if perfectly conducting walls are added perpendicularly to the magnetic field.



## TM waves ( $H_z=0$ )

$$\nabla_{xy}^2 E_z + (\gamma^2 + k^2) E_z = 0$$

Longitudinal: 2nd order PDE for  $E_z$ . we defer solution until we have defined a geometry plus b/c.

Transverse solutions once  $E_z$  is found

$$H_x = \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial y}$$

$$H_y = -\frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x}$$

$$E_x = -\frac{\gamma}{k_c^2} \frac{\partial E_z}{\partial x}$$

$$E_y = -\frac{\gamma}{k_c^2} \frac{\partial E_z}{\partial y}$$

# Further Simplification

The two E-components can be combined. If we use the notation:

$$\vec{E}_t = E_x \hat{x} + E_y \hat{y} = -\frac{\gamma}{k_c^2} \nabla_{xy} E_z \quad \text{where} \quad \nabla_{xy} = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y}$$

$$E_t = -\frac{\gamma}{k_c^2} \nabla_{xy} E_z$$

$$Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\gamma}{j\omega\epsilon} \Omega$$

$$\vec{H} = \frac{\hat{z} \times \vec{E}}{Z_{TM}}$$

# Eigenvalues

We will discover that in closed systems, solutions are possible only for discrete values of  $k_c$ . There may be an infinity of values for  $k_c$ , but solutions are not possible for all  $k_c$ . Thus  $k_c$  are known as eigenvalues. Each eigenvalue will determine the properties of a particular TM **mode**. The eigenvalues will be geometry dependent.

Assume for the moment we have determined an appropriate value for  $k_c$ , we now wish to determine the propagation conditions for a particular mode.

## Advanced Photon Source

We have the following propagation vector components for the modes in a rectangular wave guide

$$\beta^2 = \omega^2 \mu \epsilon = \beta_x^2 + \beta_y^2 + \beta_z^2$$

$$\beta_x = \frac{m\pi}{a}; \beta_y = \frac{n\pi}{a}$$

$$\beta_z^2 = \left(\frac{2\pi}{\lambda_z}\right)^2 = \left(\frac{2\pi}{\lambda_g}\right)^2 = \omega^2 \mu \epsilon - \beta_x^2 - \beta_y^2$$

$$\beta_z^2 = \omega^2 \mu \epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

At the cut-off, we have

$$\beta_z^2 = 0 = (2\pi f_c)^2 \mu \epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

## Advanced Photon Source

The cut-off frequencies for all modes are

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2}$$

With cut-off wavelengths

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2}}$$

With indices

TE modes  $m=0,1,2,3,\dots$

$n=0,1,2,3,\dots$

TM modes  $m=1,2,3,\dots$

$n=1,2,3,\dots$

(but  $m=n=0$  not allowed)

## Cut-off

Since the wave propagates according to  $e^{\pm\gamma z}$ . Then propagation ceases when  $\gamma = 0$ .

since  $\gamma = \sqrt{k_c^2 - \omega^2 \mu \epsilon}$  then  $\gamma = 0$  implies  $\omega_c^2 \mu \epsilon = k_c^2$

Or

$$f_c = \frac{k_c}{2\pi \sqrt{\mu \epsilon}}$$

Cut-off  
frequency

## Write $\gamma$ in terms of $f_c$

It is usual, now to write  $\gamma$  in terms of the cut-off frequency. This allows us to physically interpret the result.

$$\gamma = \sqrt{k_c^2 - \omega^2 \mu \epsilon} = \sqrt{k_c^2 - \frac{\omega^2 k_c^2}{\omega_c^2}} = k_c \sqrt{1 - \frac{f^2}{f_c^2}}$$

This part from the definition see slide 8/5.

Substitute for  $\mu \epsilon$  from definition of  $f_c$

Recall similarity of this result with  $\beta$  for an ionized gas. see slide 7/12

$$\gamma^2 = k_c^2 - k^2$$

# Conditions for Propagation

There are two possibilities here:

①  $f > f_c$   $\longrightarrow$   $\gamma$  is imaginary

with  $\gamma = j\beta = j\sqrt{k^2 - k_c^2}$

This says now that  $\gamma$  becomes  $j\beta$  with  
 $\beta = \sqrt{k^2 - k_c^2}$

$$= jk\sqrt{1 - \frac{k_c^2}{k^2}}$$

$$= jk\sqrt{1 - \frac{f_c^2}{f^2}}$$

this is a special case of the result in the previous slide

We conclude that if the operational frequency is above cut-off then the wave is propagating with the form  $e^{-j\beta z}$

# Different wavelengths

The corresponding wavelength inside the guide is

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - \frac{f_c^2}{f^2}}} > \lambda$$

g for guide

This is the “free space” wavelength

The free space wavelength may be written alternatively

$$\lambda = \frac{2\pi}{k}$$

Now if we introduce a cut-off wavelength  $\lambda_c = v/f_c$  where  $v$  is the corresponding velocity ( $=c$ , in air) in an unbounded medium. We can derive:

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}$$

# Dispersion in waveguides

The previous relationship showed that  $\beta$  was a function of frequency i.e. waveguides are **dispersive**. Hence we expect the phase velocity to also be a function of frequency. In fact:

$$v_p = \frac{\omega}{\beta} = \frac{v}{\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{\lambda_g}{\lambda} v > v$$

← This can be  $> c!$

So, as expected the phase velocity is always higher than in an unbounded medium (fast wave) and is frequency dependent. So we conclude waveguides are dispersive.

# Group velocity

This is similar to as discussed previously.

$$v_g = \frac{1}{\frac{\partial \beta}{\partial \omega}} = v \sqrt{1 - \frac{f_c^2}{f^2}} = \frac{\lambda}{\lambda_g} v < v$$

So the group velocity is always less than in an unbounded medium. And if the medium is free space then  $v_g v_p = v^2 = c^2$  which is also as previously discussed. Finally, recall that the energy transport velocity is the group velocity.

# Wave Impedance

Wave impedance can also be written in terms of the radical:

$$\sqrt{1 - \frac{f_c^2}{f^2}}$$

For  $f > f_c$  Then the impedance is real and less than the surrounding medium dielectric

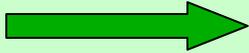
In particular:

$$\frac{\gamma}{j\omega\epsilon} = \frac{k\sqrt{1 - \frac{f_c^2}{f^2}}}{\omega\epsilon} = Z_{TM} = \eta\sqrt{1 - \frac{f_c^2}{f^2}}$$

The factor  $k/\omega\epsilon$  can be shown to be:

$$\sqrt{\frac{\mu_0}{\epsilon}} = \eta = 377\Omega \text{ (if air)}$$

# Evanescent waves

②  $f < f_c$    $\gamma$  is real

$$\text{with } \gamma = \alpha = k \sqrt{1 - \frac{k_c^2}{k^2}} = k \sqrt{1 - \frac{f_c^2}{f^2}}$$

We conclude that the propagation is of the form  $e^{-\alpha z}$  i.e. the wave is attenuating or is **evanescent** as it propagates in the  $+z$  direction. This is happening for frequencies below the cut-off frequency. At  $f=f_c$  the wave is said to be cut-off. Finally, note that there is no loss mechanism contributing to the attenuation.

## Impedance for evanescent waves

A similar derivation to that for the propagating case produces:

$$Z_{TM} = -j \frac{k_c}{\omega \epsilon} \sqrt{1 - \frac{f^2}{f_c^2}}$$

This says that for TM waves, the wave impedance is capacitive and that no power flow occurs if the frequency is below cut-off. Thus evanescent waves are associated with reactive power only.

## TE Waves

A completely parallel treatment can be made for the case of TE propagation,  $E_z = 0, H_z \neq 0$ . We only give the parallel results.

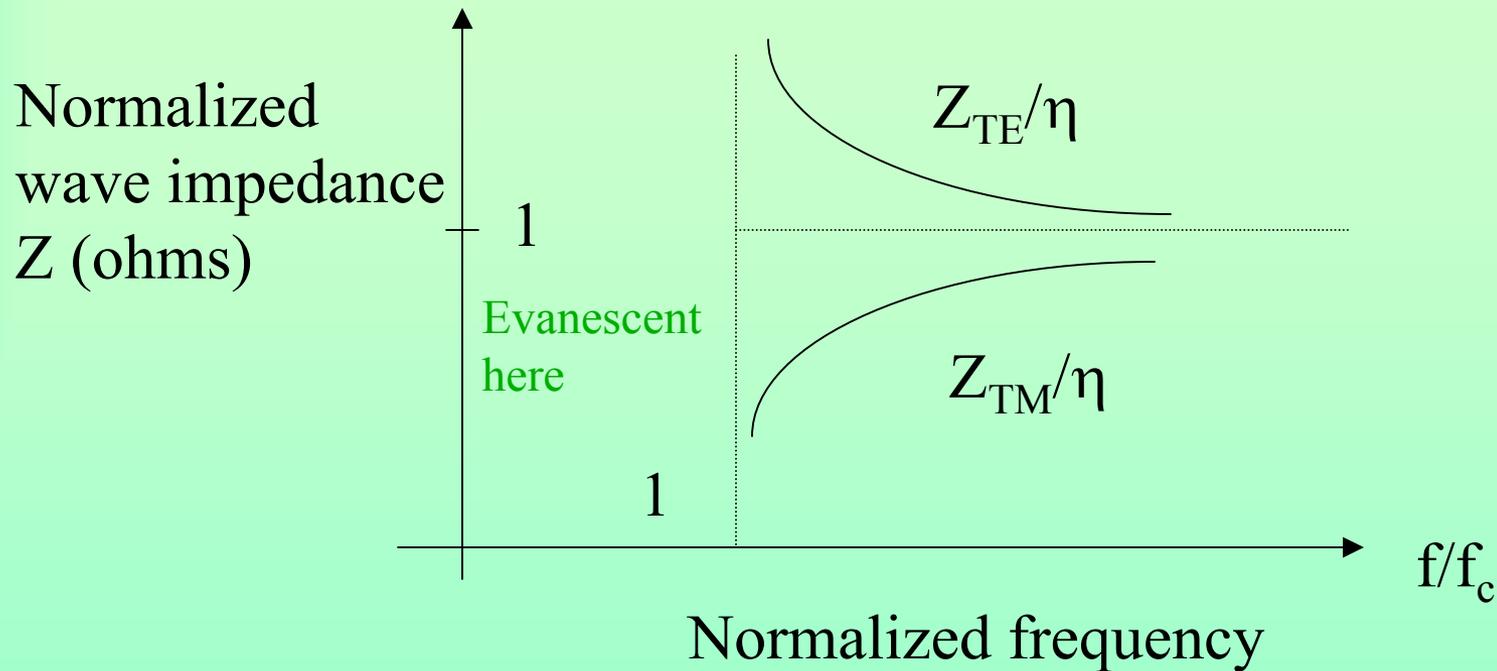
$$\nabla_{xy}^2 H_z + (\gamma^2 + k^2) H_z = 0$$

$$(H_t)_{TE} = -\frac{\gamma}{k_c^2} \nabla_{xy} H_z$$

$$Z_{TE} = \frac{j\omega\mu}{\gamma} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

$$\vec{E} = -Z_{TE} (\hat{z} \times \vec{H})$$

# Wave Impedance



# Dispersion

For propagating modes ( $\gamma = j\beta$ ), we may graph the variation of  $\beta$  with frequency (for either TM or TE) and this determines the dispersion characteristic.

$$\beta = k \sqrt{1 - \frac{f_c^2}{f^2}} \quad \text{where } v \text{ is the velocity in the unbounded medium}$$

or alternatively  $\omega = \frac{\beta v}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}}$

← This is more useful form for plotting

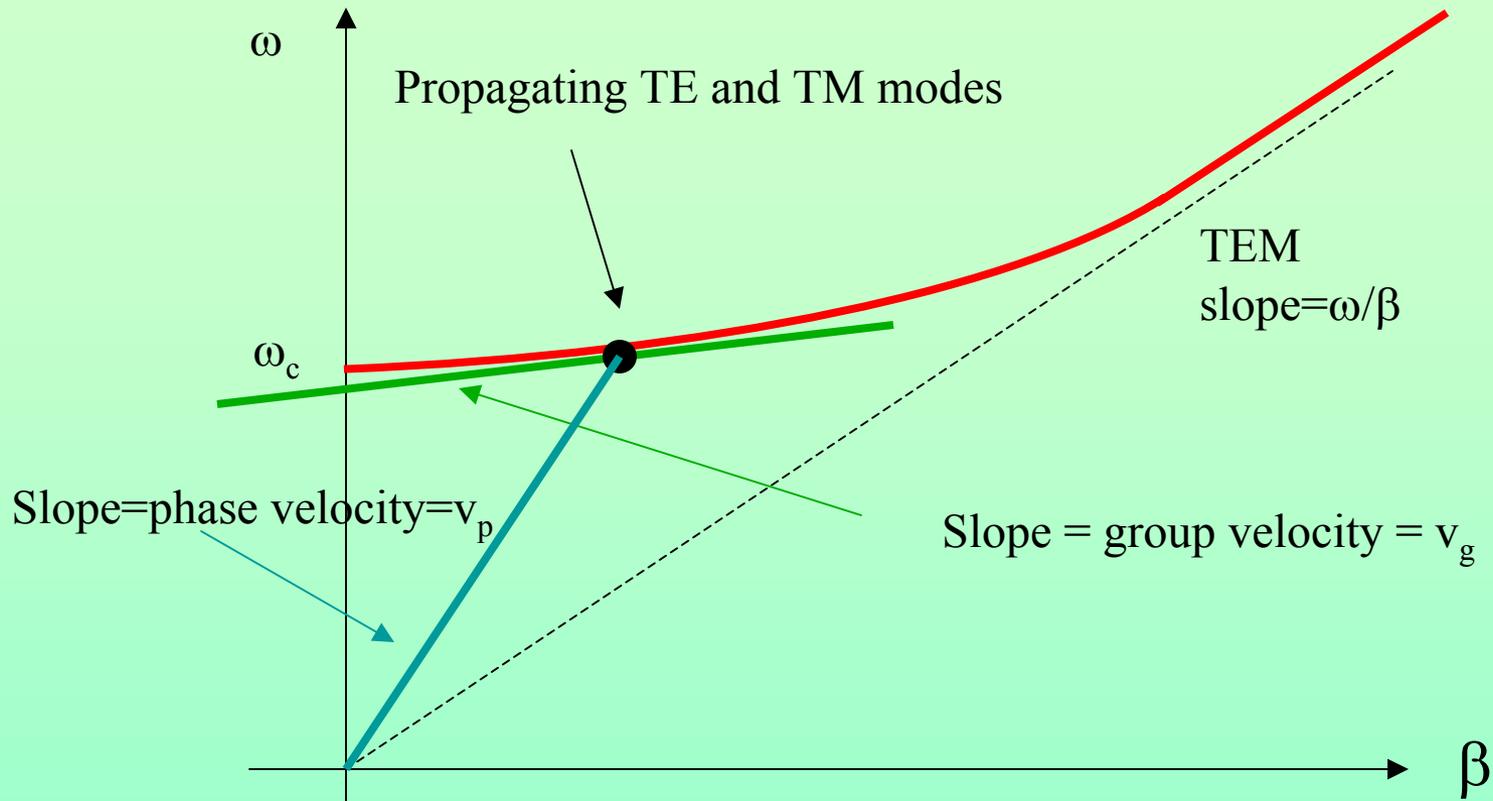
Equation of red plot

Note that  $v_p > v$

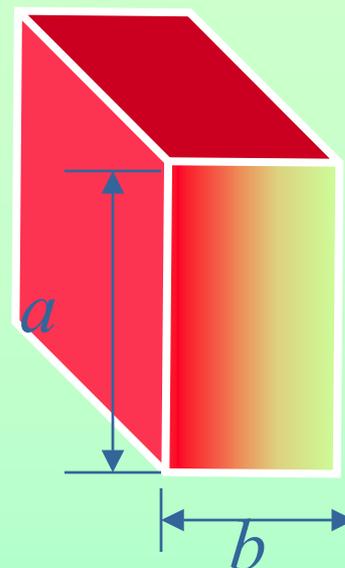
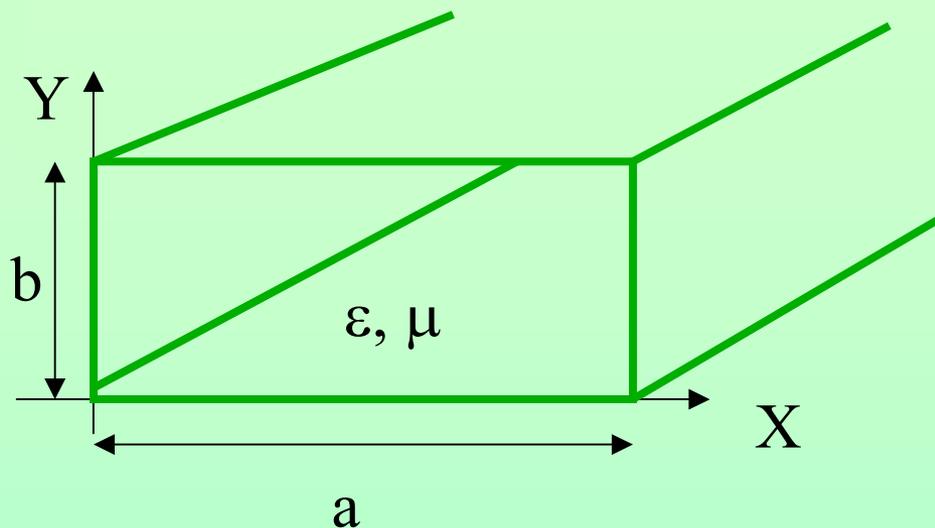
$v_g < v$

$$v_p v_g = v^2$$

# Dispersion for Waveguide



# Rectangular waveguide



Assume **perfectly conducting walls** and **perfect dielectric** filling the wave guide.

Convention always says that  $a$  is the long side.

## TM waves

TM waves have  $E_z \neq 0$ . We write  $E_z(x,y,z)$  as  $E_z(x,y)e^{-\gamma z}$ . The wave equation to solve is then

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) E_z(x, y) = 0$$

Plus some boundary conditions on the walls of the waveguide. The standard method of solving this PDE is to use separation of variables. I.e..

$$E_z(x, y) = X(x)Y(y)$$

## Possible Solutions

If we substitute into the original equation we get two more equations. But this time we have full derivatives and we can easily write solutions.  $\frac{d^2 X}{dx^2} + k_x^2 X = 0$  and  $\frac{d^2 Y}{dy^2} + k_y^2 Y = 0$

with  $k_x^2 + k_y^2 = k_c^2$

Mathematics tells us that the solutions depend on the sign of  $k_x^2$

$k_x^2$	$k_x$	<u>Appropriate X(x)</u>	
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx;$	$C_1 e^{jkx} + D_1 e^{-jkx}$
-	jk	$A_2 \sinh kx + B_2 \cosh kx;$	$C_2 e^{kx} + D_2 e^{-kx}$

# Boundary conditions

Boundary conditions say that the tangential components of  $E_z$  vanish on the walls of the guide :

$$\left. \begin{array}{l} E_z(0, y) = 0 \\ E_z(a, y) = 0 \end{array} \right\} \text{left and right hand walls.}$$

$$\left. \begin{array}{l} E_z(x, 0) = 0 \\ E_z(x, b) = 0 \end{array} \right\} \text{top and bottom walls.}$$

We choose the *sin/cos* form (why?) and directly write:

$$E_z(x, y) = (A_1 \mathbf{sin} k_x x + B_1 \mathbf{cos} k_x x)(A_2 \mathbf{sin} k_y y + B_2 \mathbf{cos} k_y y)$$

# Final solution

Using the boundary conditions, we find:

X(x) must be in the form  $\sin k_x x$

Y(y) must be in the form  $\sin k_y y$

$$\left. \begin{aligned} k_x &= \frac{m\pi}{a} \\ k_y &= \frac{n\pi}{b} \end{aligned} \right\}$$

with m, n integer and  $m, n = 1, 2, 3, \dots$

Do not start from 0

This satisfies all the boundary conditions

$$E_z(x, y) = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$k_c^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2$$

We can only get discrete values of  $k_c$  - eigenvalues!

## Mode numbers (m,n)

The  $m, n$  numbers will give different solutions for  $E_z$  (as well as all the other transverse components). Each  $m, n$  combination will correspond to a mode which will satisfy all boundary and wave equations. Notice how the modes depend on the geometry ( $a, b$ )!

We usually refer to the modes as  $TM_{mn}$  or  $TE_{mn}$  eg  $TM_{2,3}$   
Thus each mode will specify a unique field distribution in the guide. We now have a formula for the parameter  $k_c$  once we specify the mode numbers.

The concept of a mode is fundamental to many E/M problems.

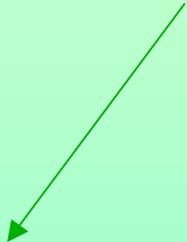
## Mode cut-off

From previous formulas, we have directly upon using the value  $k_c$

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

Note this!



Neither  $m$  nor  $n$  can be zero therefore the TM mode with the lowest cut-off frequency is  $\text{TM}_{11}$

# TE Modes

For TE modes, we have  $E_z = 0$ ,  $H_z \neq 0$  as before.

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) H_z(x, y) = 0$$

Boundary conditions

$$\left. \begin{aligned} \frac{\partial H_z}{\partial x} \Big|_{x=0} &\Rightarrow E_y = 0 \text{ at } x = 0 \\ \frac{\partial H_z}{\partial x} \Big|_{x=a} &\Rightarrow E_y = 0 \text{ at } x = a \\ \frac{\partial H_z}{\partial y} \Big|_{y=0} &\Rightarrow E_x = 0 \text{ at } y = 0 \\ \frac{\partial H_z}{\partial y} \Big|_{y=b} &\Rightarrow E_x = 0 \text{ at } y = b \end{aligned} \right\}$$

Boundary conditions for  $H_z$  (longitudinal) are equivalently expressed in terms of  $E_x$  and  $E_y$  (transverse)

# TE<sub>mn</sub> Results

The expressions for  $f_c$  and  $\lambda_c$  are identical to the TM case. But this time we have that the TE dominant mode (ie. the TE mode with the lowest cut-off frequency) is TE<sub>10</sub>. This mode has an even lower cut-off frequency than TM<sub>11</sub> and is said to be the **Dominant Mode** for a rectangular waveguide.

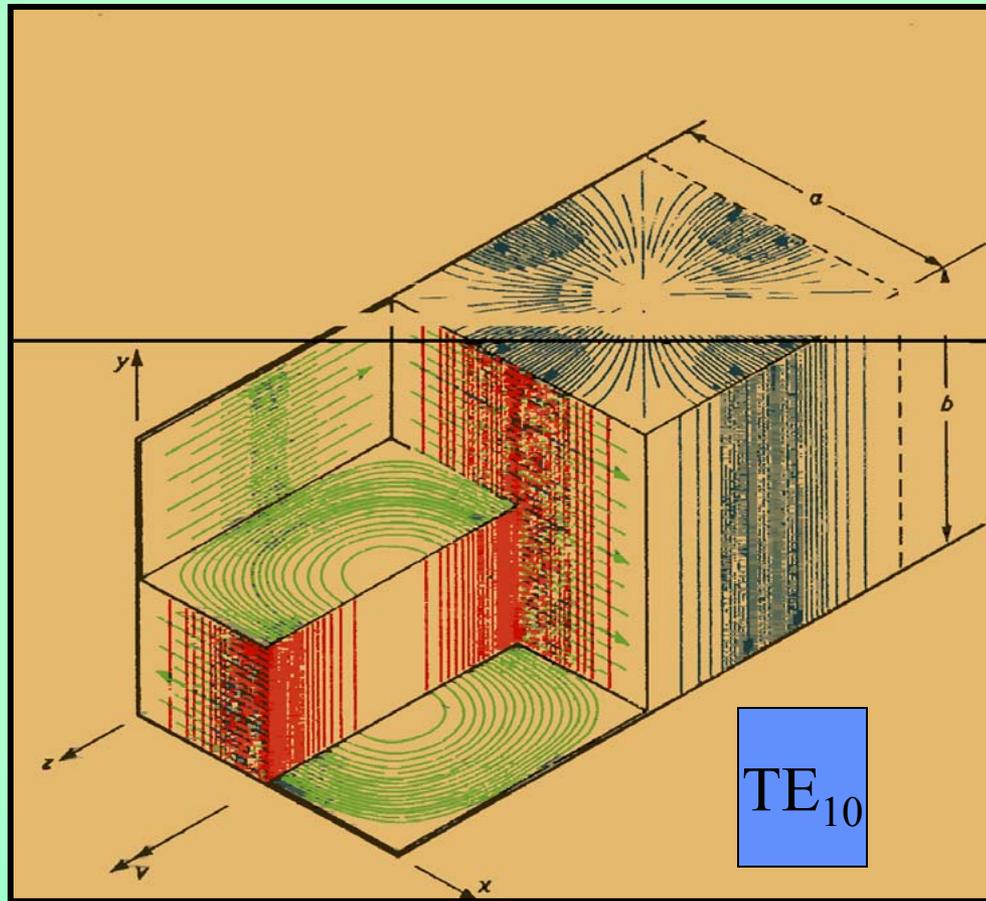
$$H_z(x, y) = H_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$(f_c)_{TE_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{v}{2a}$$

$$(\lambda_c)_{TE_{10}} = 2a$$

This is provided we label the large side 'a' and associate this side with the mode number 'm'

# View of TE<sub>10</sub> mode for waveguide.



-  H field
-  Current
-  E field

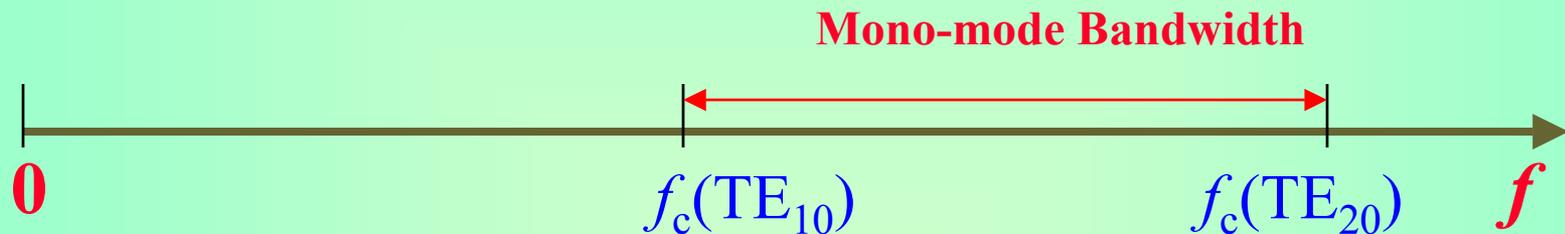
For **mono-mode** (or **single-mode**) operation, only the fundamental **TE<sub>10</sub>** mode should be propagating over the frequency band of interest.

The **mono-mode bandwidth** depends on the cut-off frequency of the **second** propagating mode. We have two possible modes to consider, **TE<sub>01</sub>** and **TE<sub>20</sub>**.

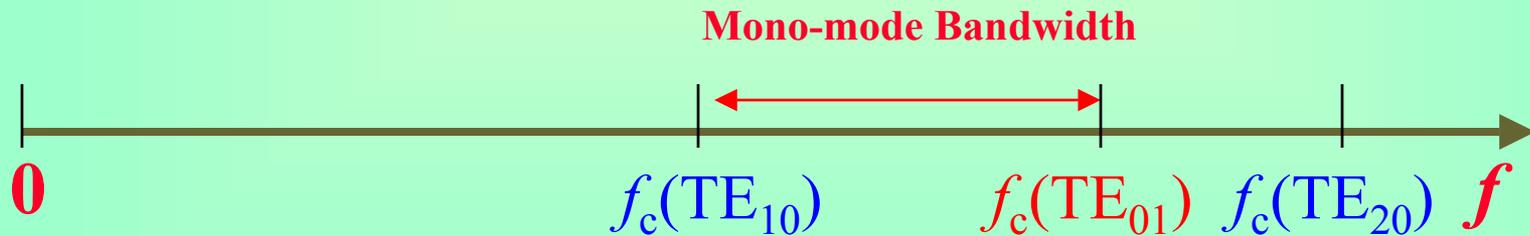
$$f_c(TE_{01}) = \frac{1}{2b\sqrt{\mu\varepsilon}}$$

$$f_c(TE_{20}) = \frac{1}{a\sqrt{\mu\varepsilon}} = 2f_c(TE_{10})$$

$$\mathcal{N} \text{ } b = \frac{a}{2} \Rightarrow f_c(TE_{01}) = f_c(TE_{20}) = 2f_c(TE_{10}) = \frac{1}{a\sqrt{\mu\epsilon}}$$



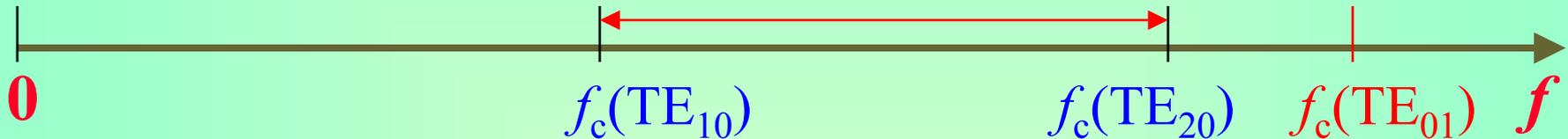
$$\mathcal{N} \text{ } a > b > \frac{a}{2} \Rightarrow f_c(TE_{10}) < f_c(TE_{01}) < f_c(TE_{20})$$



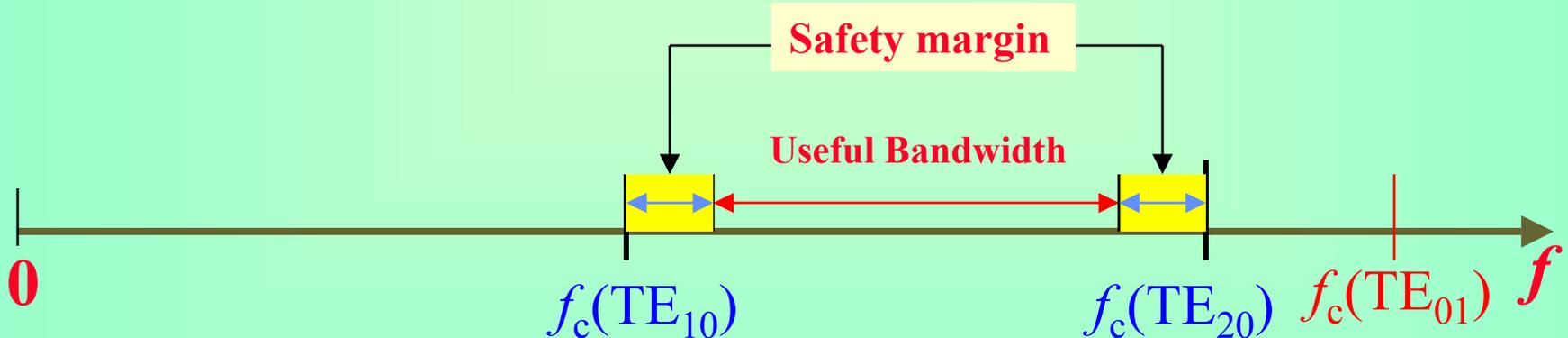
## Advanced Photon Source

$$\text{If } b < \frac{a}{2} \Rightarrow f_c(\text{TE}_{20}) < f_c(\text{TE}_{01})$$

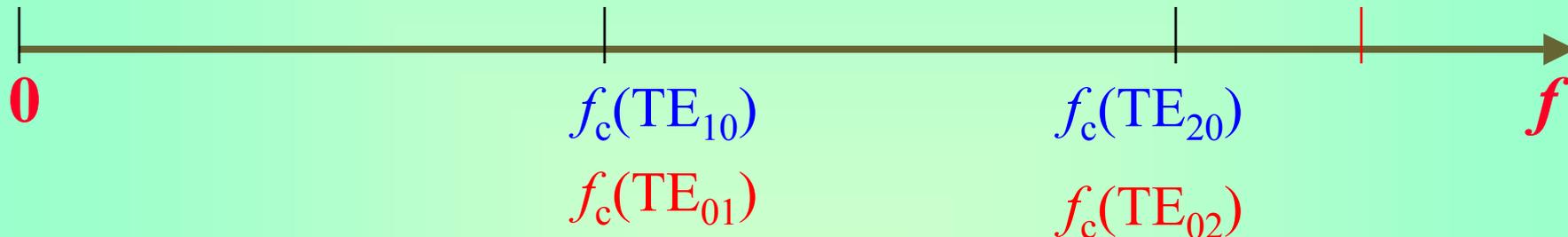
**Mono-mode Bandwidth**



In practice, a **safety margin** of about **20%** is considered, so that the **useful bandwidth** is less than the maximum mono-mode bandwidth. This is necessary to make sure that the first mode ( $\text{TE}_{10}$ ) is well **above cut-off**, and the second mode ( $\text{TE}_{01}$  or  $\text{TE}_{20}$ ) is strongly **evanescent**.



If  $a=b$  (square wave guide)  $\Rightarrow f_c(TE_{10}) = f_c(TE_{20})$



In the case of perfectly square wave guide,  $TE_{m0}$  and  $TE_{0n}$  modes with  $m=n$  are **degenerate** with the same cut-off frequency.

Except for **orthogonal** field orientation, all other properties of the degenerate modes are the same.

**Example** – Design an air-filled rectangular wave guide for the following operation conditions:

- 10 GHz in the middle of the frequency band (single mode operation)
- $b=a/2$

The fundamental mode is the  $TE_{10}$  with cut-off frequency

$$f_c(TE_{10}) = \frac{1}{2a\sqrt{\mu_0\epsilon_0}} = \frac{c}{2a} \approx \frac{3 \times 10^8 \text{ m/sec}}{2a} \text{ Hz}$$

For  $b=a/2$ ,  $TE_{01}$  and  $TE_{20}$  have the same cut-off frequency

$$f_c(TE_{01}) = \frac{1}{2b\sqrt{\mu_0\epsilon_0}} = \frac{c}{2b} = \frac{c}{2a} = \frac{c}{2a} \approx \frac{3 \times 10^8 \text{ m/sec}}{a} \text{ Hz}$$

$$f_c(TE_{20}) = \frac{1}{a\sqrt{\mu_0\epsilon_0}} = \frac{c}{a} \approx \frac{3 \times 10^8 \text{ m/sec}}{a} \text{ Hz}$$

The operation frequency can be expressed in terms of the cut-off frequencies

$$\begin{aligned} f &= f_c(TE_{10}) + \frac{f_c(TE_{10}) - f_c(TE_{01})}{2} \\ &= \frac{f_c(TE_{10}) + f_c(TE_{01})}{2} = 10.0 \text{GHz} \end{aligned}$$

$$\Rightarrow 10.0 \times 10^9 = \frac{1}{2} \left[ \frac{3 \times 10^8}{2a} + \frac{3 \times 10^8}{a} \right]$$

$$\Rightarrow a = 2.25 \text{cm} \quad b = \frac{a}{2} = 1.125 \text{cm}$$

## Sample waveguide calculation

We consider an air filled guide, so  $\epsilon_r=1$ . The internal size of the guide is 0.9 x 0.4 inches (waveguides come in standard sizes). The cut-off frequency of the dominant mode:

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad a = 0.9'' = 22.86\text{mm}; b = 0.4'' = 10.16\text{mm}$$

$$(k_c)_{TE_{10}} = \frac{\pi}{a} = 137.43$$

$$(f_c)_{TE_{10}} = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}} = \frac{137.43 \times 3 \times 10^8}{2\pi} = 6.56\text{GHz}$$

## Sample calculation (ctd)

The next few modes are:

$$\left. \begin{array}{l} (k_c)_{11} = 338.38 \\ (k_c)_{01} = 309.21 \\ (k_c)_{20} = 274.86 \end{array} \right\} \text{the ascending order of mode is } 10, 20, 01, 11$$

The next cutoff frequency after  $TE_{10}$  will then be

$$(f_c)_{TE_{20}} = \frac{274.86 \times 3 \times 10^8}{2\pi} = 13.12 \text{GHz}$$

So for **single mode operation** we must operate the guide within the frequency range of  $6.56 < f < 13.12 \text{GHz}$ .

## Sample Calculation (ctd)

It is not good to operate too close to cut-off for the reason that the wall losses increases very quickly as the frequency approaches cut-off. A good guideline is to operate between  $1.25f_c$  and  $1.9f_c$ . This then would restrict the single mode operation to 8.2 to 12.5 GHz.

The propagation coefficient for the next higher mode is:

$$(\gamma)_{20} = \sqrt{k_c^2 - k^2} = (k_c)_{20} \sqrt{1 - \frac{f^2}{f_{c20}^2}}$$

Specify an operating frequency  $f$ , half way in the original range of  $TE_{10}$  i.e.. 9.84GHz.

## Sample Calculation (ctd)

$$(\gamma)_{20} = 274.86 \sqrt{1 - \left(\frac{9.84}{13.12}\right)^2} = 181.8 \text{ (Real)}$$

So  $\alpha = 181.8 \text{ Np/m}$

or in dB  $181.8 \times 8.7 = 1581 \text{ dB/m}$  ie.  $\text{TE}_{20}$  is very strongly evanescent.

In comparison for  $\text{TE}_{10}$ :

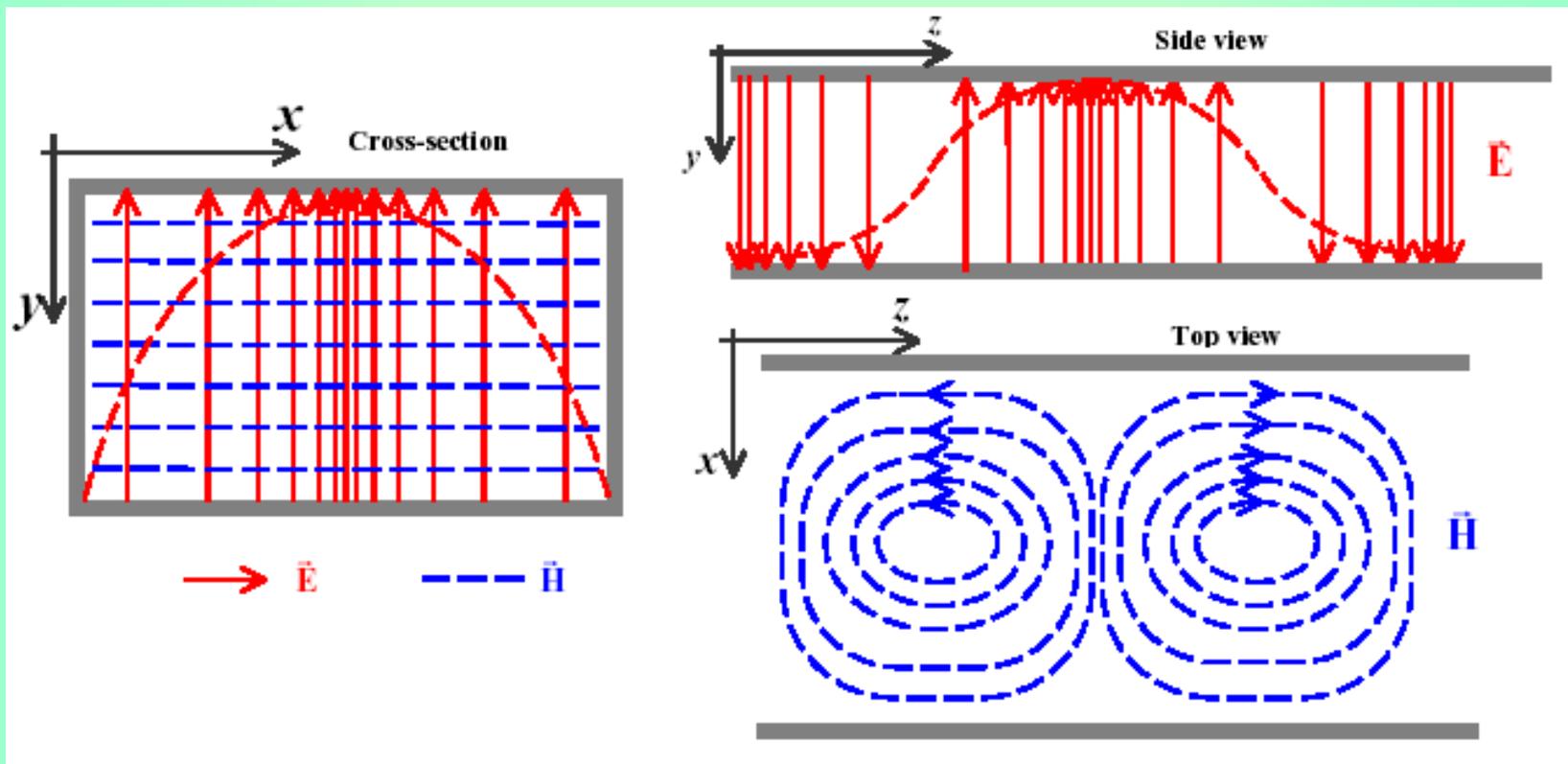
$$(\gamma)_{10} = (k_c)_{10} \sqrt{1 - \frac{f^2}{f_{c10}^2}} = 137.43 \sqrt{1 - \left(\frac{9.84}{6.56}\right)^2} = 153.64j \text{ (Imaginary)}$$

So  $\beta = 153.64 \text{ rad/m}$

All further higher order modes will be cut-off with higher rates of attenuation.

# Field Patterns

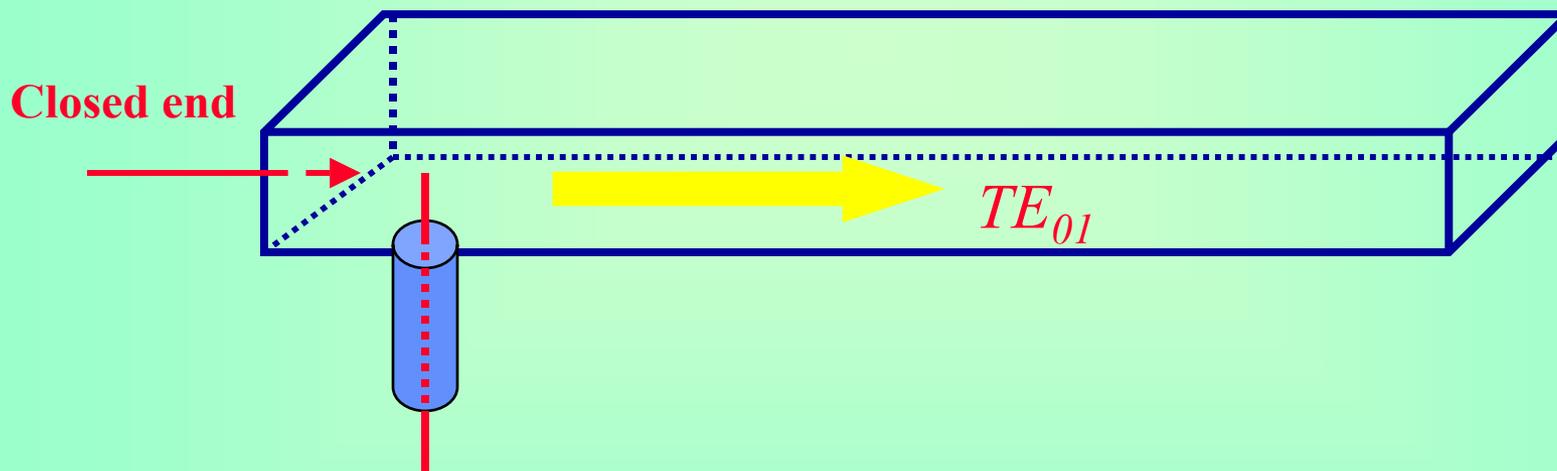
Field patterns for the  $TE_{10}$  mode in rectangular wave guide



## Advanced Photon Source

The simple arrangement below can be used to excite  $TE_{10}$  in a rectangular wave guide.

The inner conductor of the coaxial cable behaves like a dipole antenna and it creates a maximum electric field in the middle of the cross-section.



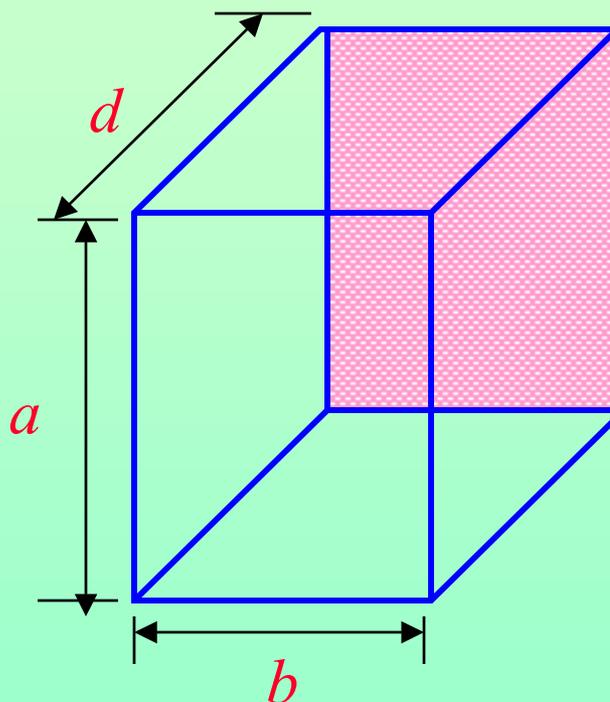
# Waveguide Cavity Resonator

The **cavity resonator** is obtained from a section of rectangular wave guide, closed by two additional metal plates. We assume again **perfectly conducting walls** and **loss-less dielectric**.

$$\beta_x = \frac{m\pi}{a}$$

$$\beta_y = \frac{m\pi}{b}$$

$$\beta_z = \frac{p\pi}{d}$$



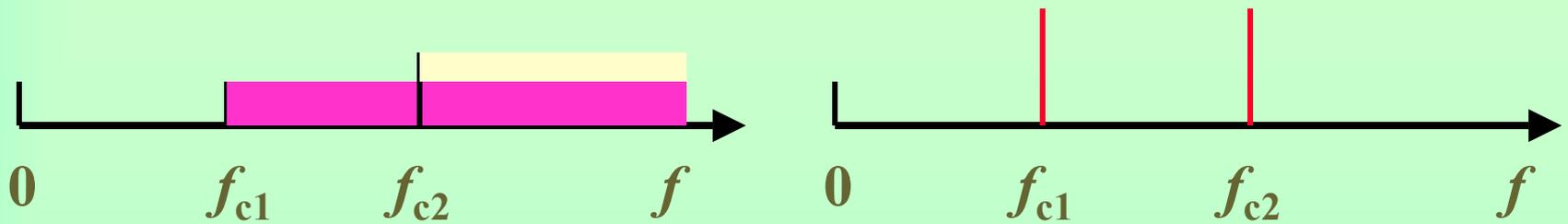
# Waveguide Cavity Resonator

The addition of a new set of plates introduces a condition for **standing waves** in the z-direction which leads to the definition of oscillation frequencies

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

The **high-pass** behavior of the rectangular wave guide is modified into a **very narrow pass-band** behavior, since cut-off frequencies of the wave guide are transformed into **oscillation frequencies** of the resonator.

# Waveguide Cavity Resonator



In the wave guide, each mode is associated with a band of frequencies larger the cut-off frequency.

In the resonator, resonant modes can only exist in correspondence of discrete resonance frequencies.

# Waveguide Cavity Resonator

The cavity resonator will have modes indicated as

$$TE_{mnp}$$

$$TM_{mnp}$$

The values of the index corresponds to periodicity (number of sine or cosine waves) in three direction. Using z-direction as the reference for the definition of transverse electric or magnetic fields, the allowed indices are

$$TE \begin{cases} m = 0,1,2,3,\dots \\ n = 0,1,2,3,\dots \\ p = 0,1,2,3,\dots \end{cases}$$

$$TM \begin{cases} m = 0,1,2,3,\dots \\ n = 0,1,2,3,\dots \\ p = 0,1,2,3,\dots \end{cases}$$

With only one zero index m or n allowed

The mode with lowest resonance frequency is called **dominant mode**. In case  $a \geq d > b$  the dominant mode is the  $TE_{101}$ .

# Waveguide Cavity Resonator

Note that a **TM** cavity mode, with magnetic field transverse to the  $z$ -direction, it is possible to have the **third index** equal **zero**. This is because the magnetic field is going to be parallel to the third set of plates, and it can therefore be uniform in the third direction, with no periodicity.

The **electric field** components will have the following form that satisfies the **boundary conditions** for perfectly conducting walls.

$$E_x = \mathcal{E}_x \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = \mathcal{E}_y \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right)$$

$$E_z = \mathcal{E}_z \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right)$$

# Waveguide Cavity Resonator

The amplitudes of the **electric field** components also must satisfy the divergence condition which, in absence of charge is

$$\nabla \cdot \vec{E} = 0 \Rightarrow \left( \frac{m\pi}{a} \right) E_x + \left( \frac{m\pi}{b} \right) E_y + \left( \frac{p\pi}{d} \right) E_z = 0$$

The **magnetic field** intensities are obtained from **Ampere's law**:

$$H_x = \frac{\beta_z E_y - \beta_y E_z}{j\omega\mu} \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{m\pi}{b} y\right) \cos\left(\frac{p\pi}{d} z\right)$$

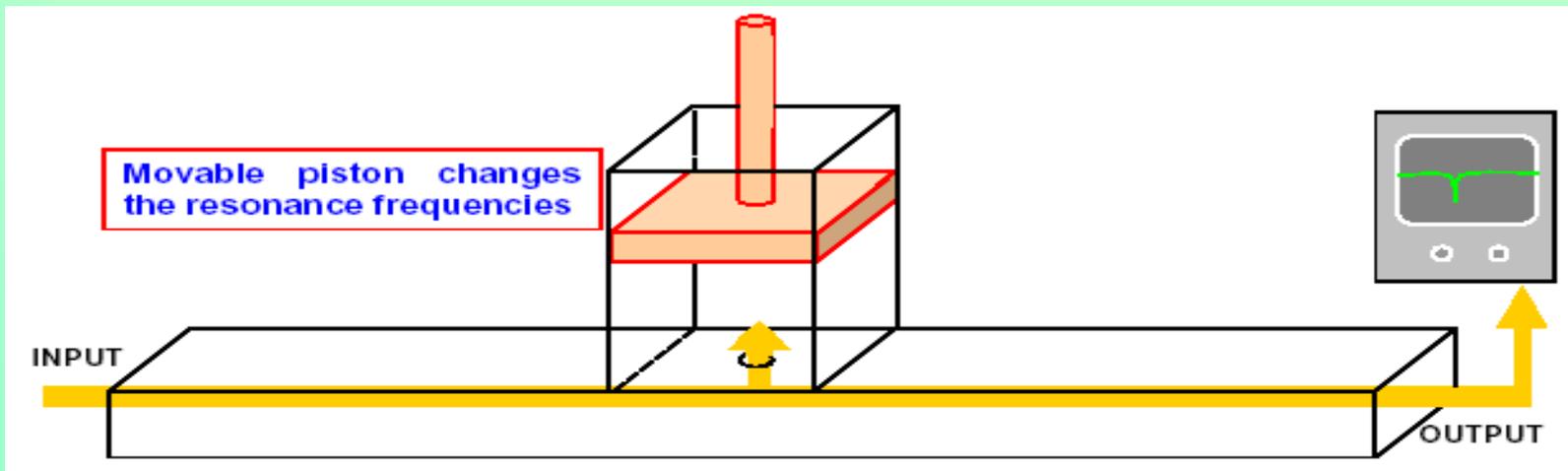
$$H_y = \frac{\beta_x E_z - \beta_z E_x}{j\omega\mu} \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \cos\left(\frac{p\pi}{d} z\right)$$

$$H_z = \frac{\beta_y E_x - \beta_x E_y}{j\omega\mu} \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{m\pi}{b} y\right) \sin\left(\frac{p\pi}{d} z\right)$$

# Waveguide Cavity Resonator

Similar considerations for **modes** and **indices** can be made if the other axes are used as a reference for the transverse field, leading to analogous resonant field configurations.

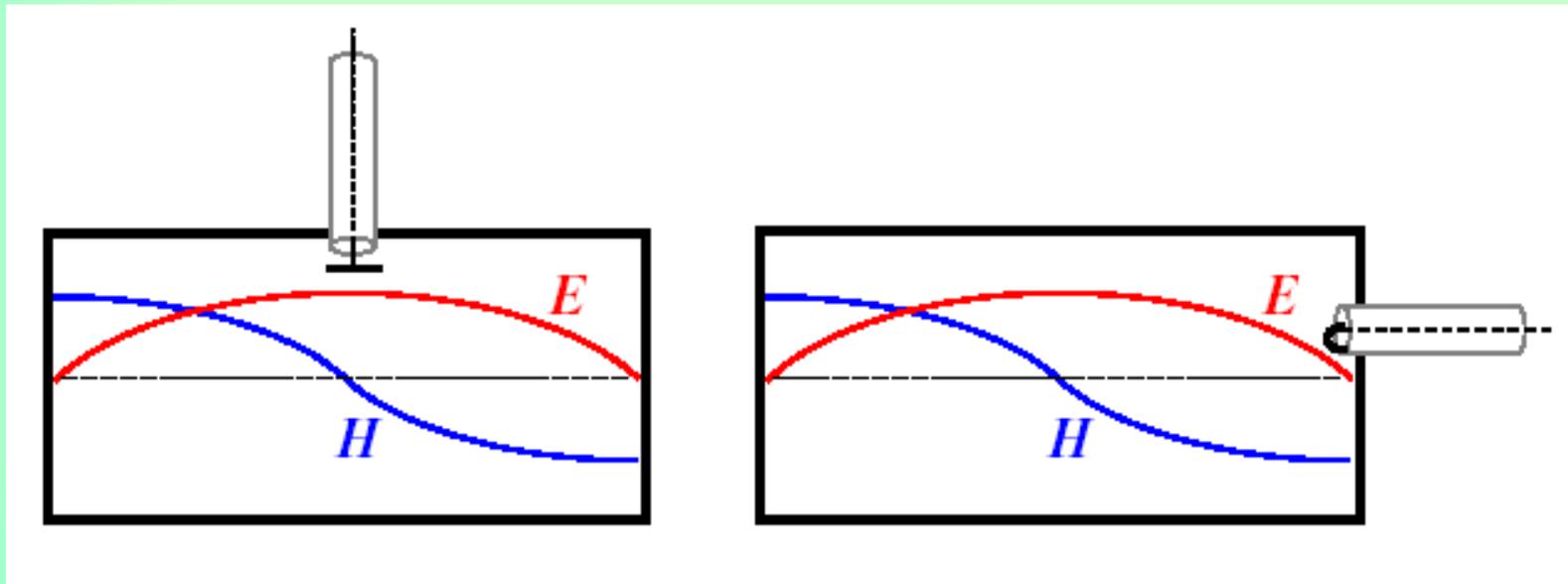
A cavity resonator can be **coupled** to a **wave guide** through a small opening. When the input frequency resonates with the cavity, electromagnetic radiation enters the resonator and a lowering in the output is detected. By using carefully tuned cavities, this scheme can be used for **frequency measurements**.



# Waveguide Cavity Resonator

Example of resonant cavity excited by using coaxial cables.

The **termination** of the **inner conductor** of the cable acts like an **elementary dipole** (left) or an **elementary loop** (right) antenna.



Excitation with a dipole antenna

Excitation with a loop antenna