LS-131
K. Symon
August 1988

Corrections in pencil per K. Symon 9-2-97 made to [illegible] copy

Analysis of a Third-Integral Resonance

In preparation for the Aladdin experiments, I will give an analytic treatment of a third-integral resonance. By starting from the equations for an actual ring, we can then connect the analytic parameters with the real ring.

1. Analysis of the Resonance.

The Hamiltonian in the neighborhood of the third-integral resonance $3\nu = m$ can be written in terms of angle-action variables J, γ in the form

$$h = \nu J - S(2J)^{3/2} \sin(3\gamma - m\theta + 3\zeta) + \frac{1}{4}a(2J)^2 , \tag{1.1}$$

where I have suppressed the subscript x on ν, J, γ, and S, a, ζ, are parameters to be connected later with the real machine. The independent variable θ runs from 0 to 2π around the ring. I have kept only the linear term, the resonance term, and a frequency shifting term. All other terms are assumed transformed away; their effects remain only in the frequency shifting coefficient a.

We first transform to a rotating coordinate system via the generating function

$$F(\underline{J},\gamma,\theta) = \underline{J}(\gamma - \frac{m}{3}\theta + \zeta) , \tag{1.2}$$

which gives

$$\underline{\gamma} = \gamma - \frac{m}{3}\theta + \zeta , \quad \underline{J} = J , \quad \underline{h} = h - \frac{m}{3}\underline{J} . \tag{1.3}$$

the new Hamiltonian is

$$\underline{h} = (\nu - \frac{m}{3}) J - S(2J)^{3/2} \sin 3\underline{\gamma} + \frac{1}{4}a (2J)^2 , \tag{1.4}$$

where J is written for $\underline{J} = J$. In rectangular canonical coordinates

$$X = (2J)^{1/2}\sin\underline{\gamma}\ ,\ P = (2J)^{1/2}\cos\ \underline{\gamma}\ , \tag{1.5}$$

$\underline{h}$ is

$$\underline{h} = \frac{1}{2}(\nu - \frac{m}{3})(X^2 + P^2) - 3\ SXP^2 + SX^3 + \frac{1}{4}a(X^2+P^2)^2\ . \tag{1.6}$$

If we omit the term in a, then at the separatrix, Eq. (1.6) factors into the product of three straight lines:

$$S[X - \frac{\nu-m/3}{6S}][\sqrt{3}P + X + \frac{\nu-m/3}{3S}][\sqrt{3}P - X - \frac{\nu-m/3}{3S}] = 0\ . \tag{1.7}$$

The phase plane for this case is shown in Fig. 1. The limiting value of X at the separatrix is

$$X_s = (2J_S)^{1/2} = |\frac{\nu-m/3}{3S}|\ . \tag{1.8}$$

If we keep the frequency shifting term, it is easier to work with the Hamiltonian in the form (1.4). The fixed points occur where

$$\frac{\partial \underline{h}}{\partial \underline{\gamma}} = -3S(2J)^{3/2}\ \cos 3\underline{\gamma} = 0\ ,$$

and

$$\frac{\partial \underline{h}}{\partial J} = \nu - \frac{m}{3}\ 3S(2J)^{1/2}\sin 3\underline{\gamma} + a(2J) = 0\ . \tag{1.9}$$

The solutions are:

$$\underline{\gamma} = \pi/6,\ 5\pi/6,\ 3\pi/2 \text{ and } \pi/2,\ 7\pi/6,\ 11\pi/6\ ,$$

$$(2J)^{1/2} = \frac{(\pm)3S}{2a} \pm [(\frac{3S}{2a})^2 - \frac{\nu-m/3}{a}]^{1/2}\ , \tag{1.10}$$

where the upper sign in (±) refers to the first three solutions for $\underline{\gamma}$, and the lower, to the second three. There are three cases:

a) $S^2 < (4/9)(\nu - m/3)a$ [$(\nu - m/3)$ and a have the same sign].
No roots. Phase plane shown in Fig. 2.

b) $S^2 > (4/9)(\nu - m/3)a > 0$.

Two roots; we must choose $\underline{\gamma}$ so $(\pm)S/a > 0$. Separatrix has ears (Fig. 3).

$$X_S = (2J_S)^{1/2} = \left|\frac{3S}{2a}\right| - \left[\left(\frac{3S}{2a}\right)^2 - \frac{\nu - m/3}{a}\right]^{1/2} ,$$

$$X_0 = (2J_0)^{1/2} = \left|\frac{3S}{2a}\right| + \left[\left(\frac{3S}{2a}\right)^2 - \frac{\nu - m/3}{a}\right]^{1/2} . \tag{1.11}$$

c) $(\nu - m/3)$ and a have opposite signs.
One root for each sign in $(\pm)$. Separatrix has islands (Fig. 4).

$$X_S = (2J_S)^{1/2} = -\left|\frac{3S}{2a}\right| + \left[\left(\frac{3S}{2a}\right)^2 + \frac{m/3 - \nu}{a}\right]^{1/2} ,$$

$$X_0 = (2J_0)^{1/2} = + \left|\frac{3S}{2a}\right| + \left[\left(\frac{3S}{2a}\right)^2 + \frac{m/3 - \nu}{a}\right]^{1/2} . \tag{1.12}$$

It is not easy to calculate the sizes of the loops in cases b) and c). Let us consider the exactly resonant case, $\nu = m/3$, shown in Fig. 5. [Fig. 5 is drawn for the case $S/a > 0$.] The solution of Eq. (1.9) is

$$X_0 = (2J_0)^{1/2} = \left|\frac{3S}{a}\right| , \tag{1.13}$$

The separatrix is given by the equation

$$\underline{h} = -S(2J)^{3/2} \sin(3\underline{\gamma}) + \frac{1}{4}a(2J)^2 = 0 , \tag{1.14}$$

whose solution on the ray through the elliptic fixed point is the origin and

$$X_\ell = (2J_\ell)^{1/2} = \left|\frac{4S}{a}\right| . \tag{1.15}$$

2. Connection with the Real Ring.

We start with the Hamiltonian for the real ring (e.g., Aladdin), in terms of the variables x, y, and the independent variable s (distance measured along the central orbit):

Changes are marked per K. Symon 9-2-97

$$H_{real} = \frac{1}{2}(p_x{}^2 + P_y{}^2) + \frac{1}{2}(K_x X^2 + K_y y^2) + \frac{B''}{6B\rho}(x^3 - 3xy^2) \;, \tag{2.1}$$

where we have included linear and sextupole terms. Higher-order terms can be added without changing the treatment essentially. The momenta are defined by

$$p_z = \frac{\partial H}{\partial p_z} = \frac{dz}{ds} = z' \;, \tag{2.2}$$

where z is either x or y. We make a canonical transformation to the angle-action variables for the linear motion:

$$\begin{aligned} z &= \beta_z{}^{1/2}(2J_z)^{1/2}\sin(\gamma_z - \psi_z) \;, \\ p_z &= \beta_z^{-1/2}(2J_z)^{1/2}[\cos(\gamma_z - \psi_z) - \alpha_z\sin(\gamma_z - \psi_z)] \;, \end{aligned} \tag{2.3}$$

where $\psi_z(s)$ is defined by

$$\psi_z(s) = \int^s [\beta_z{}^{-1} - \nu_z/R]ds \;, \tag{2.4}$$

where the constant of integration is arbitrary and may be chosen to make $\langle\psi_z\rangle = 0$, and $2\pi R$ is the circumference. The resulting Hamiltonian is

$$H_{aa} = \nu_x J_x/R + \nu_y J_y/R + \frac{B''}{6B\rho}[x^3 - 3xy^2] \;, \tag{2.5}$$

where we substitute in the square brackets for x and y from Eq. (2.3). The sextupole terms may be transformed away (see Section 3 below), except for terms which drive the resonance $3\nu_x - m = 0$. The result is to add a polynomial in J_x, J_y to the linear terms in Eq. (2.5). None of the nonlinear terms in the ideal lattice drives the resonance. Let us assume they have been transformed away, and that there remains only the single sextupole which we use to excite the resonance:

$$H_f = \nu_x J_x/R + \nu_y J_y/R + (a/R)J_x^2 + 2(b/R)J_x J_y + (c/R)J_y^2$$
$$+ \frac{B''\ell}{6B\rho}\,\delta(s-s_j)[x^3 - 3xy^2]\ , \tag{2.6}$$

where the sextupole is at $s = s_j$, and we have included non-resonant terms through fourth-order in $J^{1/2}$. In case the resonance is driven by imperfections in the lattice, the last term becomes a sum over imperfection sextupoles; these terms can be handled in essentially the same way as we treat the driven sextupole below. The coefficients a, b, c define the change in tune with amplitude. The nonlinear tunes are

$$N_x = R\frac{d\gamma_x}{ds} = R\frac{\partial H_f}{\partial J_x} = \nu_x + a(2J_x) + b(2J_y)\ ,$$

$$N_y = R\frac{d\gamma_y}{ds} = R\frac{\partial H_f}{\partial J_y} = \nu_y + b(2J_y) + c(2J_y)\ , \tag{2.7}$$

where $2J_z = z_{max}^2/\beta_z$ is the square amplitude of the motion.

To simplify the algebra, we now set y = 0, and consider only the x-motion near the resonance. The Hamiltonian is

$$H_{fr} = \nu_x J_x/R + (a/R)J_x^2 + \frac{B''\ell}{6B\rho}\,\delta(s-s_j)(2J_x)^{3/2}\beta_x^{3/2}\sin^3(\gamma_x - \psi_x)$$

$$= \nu_x J_x/R+(a/R)J_x^2+ \frac{B''\ell^{3/2}}{6B\rho}\beta_x\,\delta(s-s_j)(2J_x)^{3/2}[-\tfrac{1}{4}\sin 3(\gamma_x-\psi_x)+\tfrac{3}{4}\sin(\gamma_x-\psi_x)]\ . \tag{2.8}$$

We may expand the periodic delta-function in a Fourier series:

$$\delta(s - s_j) = \sum_{m=-\infty}^{\infty} e^{im(s-s_j)}/2\pi R\ . \tag{2.9}$$

It will also be convenient to change to an independent variable θ, which increases by 2π around the ring:

$$s = \overset{\theta R}{\cancel{\theta/R}} \ . \tag{2.10}$$

It is easy to see that the effect is to multiply the Hamiltonian by R. The result is

$$H_\theta = \nu_x J_x + aJ_x^{\ 2} + \sum_{m=-\infty}^{\infty} S(2J_x)^{3/2}\Big[-\sin\Big(3\gamma_x - m\theta + \frac{ms_j}{R} - 3\psi_{xj}\Big)$$
$$+ 3\sin\Big(\gamma_x - m\theta + \frac{ms_j}{R} - 3\psi_{xj}\Big)\Big] \ . \tag{2.11}$$

where

$$S = \left(\frac{\beta_x^{3/2} B''\ell}{48\pi\, B\rho}\right)_{s=s_j} \ . \tag{2.12}$$

not a subscript

All the terms except the resonant term can be transformed away (see below). The result is

$$H_{res} = \nu_x J_x - S(2J_x)^{3/2}\sin\Big(3\gamma_x - m\theta + \frac{ms_j}{R} - 3\psi_{xj}\Big) + aJ_x^{\ 2} \ , \tag{2.13}$$

where m is now the fixed value associated with the resonance (m = 22 for the Aladdin experiments). This is the Hamiltonian (1.1), with

$$\zeta = \psi_{xj} + ms_j/3R \ . \tag{2.14}$$

After transforming away the non-resonant terms, we are left with variables J, γ which differ from those in Eq. (2.3) by terms of order $J^{1/2}$ and higher. We will neglect this difference, and identify the final variables J, γ with those in Eq. (2.3). Since all transformations are canonical, the final variables J will be associated with phase curves having the same area as those associated with the original variables J.

3. Transforming the Non-Resonant Terms.

There are various ways of transforming away the non-resonant terms, all of which should give the same eventual result. We proceed as follows. Define a transformation J, $\gamma \to \underline{J}$, $\underline{\gamma}$ defined by the generating function

$$F(\underline{J},\gamma,\theta) = \underline{J}\gamma + \sum_m{}'(2\underline{J})^{3/2}[F_{m3}\cos(3\gamma - m\theta + \frac{ms_j}{R} - 3\psi_j)$$

$$+ F_{m1}\cos(\gamma - m\theta + ms_j/R - \psi_j)] , \tag{3.1}$$

where the summation is over all terms corresponding to those in the sum in Eq. (2.12) except the resonant term, and we suppress subscripts x for convenience. The resulting transformation is

$$J = \frac{\partial F}{\partial \gamma} = \underline{J} - \sum_m{}'(2\underline{J})^{3/2}[3F_{m3}\sin(3\gamma - ms_j/R - 3\psi_j)$$

$$+ F_{m1}\sin(\gamma - m\theta + ms_j/R - \psi_j)] , \tag{3.2}$$

$$\underline{\gamma} = \frac{\partial F}{\partial \underline{J}} = \gamma + \sum_m{}'3(2\underline{J})^{1/2}[F_{m3}\sin(3\gamma - m\theta + ms_j/R - 3\psi_j$$

$$+ F_{m3}\sin(\gamma - m\theta + ms_j/R - \psi_j)] . \tag{3.3}$$

The new Hamiltonian may be written

$$\underline{H}_\theta = H_\theta + \frac{\partial F}{\partial \theta} = \nu\underline{J} - S(2\underline{J})^{3/2}\sin(3\underline{\gamma} - m\theta + ms_j/R - 3\psi_j)$$

$$+ \sum_{\underline{m}'}{}'(2\underline{J})^{3/2}\sin(3\underline{\gamma} - \underline{m}'\theta + \underline{m}'s_j/R - 3\psi_j)[-S + F_{m3}(m-3\nu)]$$

$$+ \sum_{\underline{m}'}{}'(2\underline{J})^{3/2}\sin(\underline{\gamma} - \underline{m}'\theta + \underline{m}'s_j/R - \psi_j)[3S + F_{m1}(m-\nu)]$$

$$+ aJ^2 + H_4 + \text{h.o.t.} , \tag{3.4}$$

where H_4 includes the fourth-order terms, and h.o.t., the higher-order terms that come from the substitution. We eliminate the non-resonant third-order terms by setting

$$F_{m3} = S/(m - 3\nu), \; F_{m1} = -S/3(m - \nu). \tag{3.5}$$

The terms in H_4 include θ-independent terms arising from squared sines plus θ-dependent terms, which we may again transform away to higher order by the same method. There remain to fourth order in $J^{1/2}$ only the terms

$$H_{rc} = \nu\underline{J} - S(2\underline{J})^{3/2}\sin(3\underline{\gamma} - m\theta + ms_j/R - 3\psi_j) + (a + a_s)J^2, \tag{3.6}$$

where the addition to a, due to the added sextupole, is

$$a_s = 6S^2 \sum_{m'}{}' \left[\frac{3}{m'-3\nu} + \frac{1}{m'-\nu}\right] \doteq 6S^2\left[-\frac{1}{\delta\nu} + 3\delta\nu + 6\delta(3\nu)\right], \tag{3.7}$$

where $\delta\nu = \nu - m$, and m is the nearest integer to ν. Note that the resonant contribution $-1/\delta(3\nu)$ does not appear, since the resonant term is included explicitly in Eq. (3.6). The coefficient a itself arises from the sextupole terms in the lattice itself, plus any octupole terms. However, it is easier to get a from Eq. (2.7).

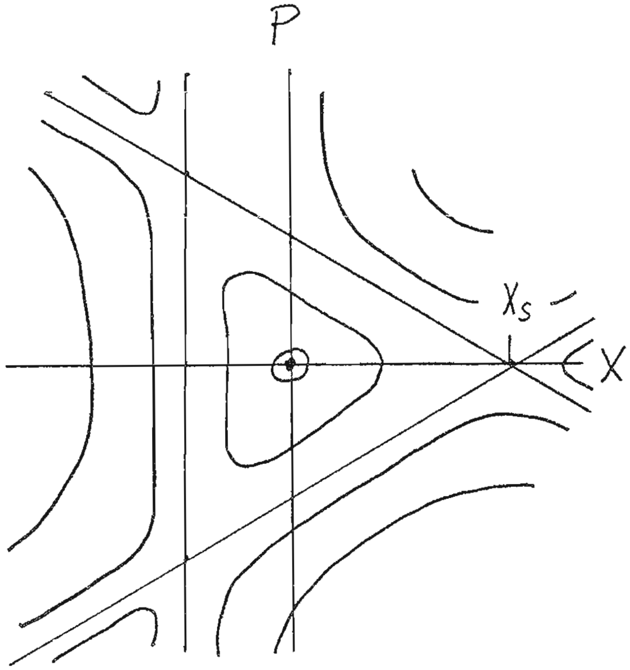

Fig. 1. Phase plane near $\nu = m/3$, a-term neglected.

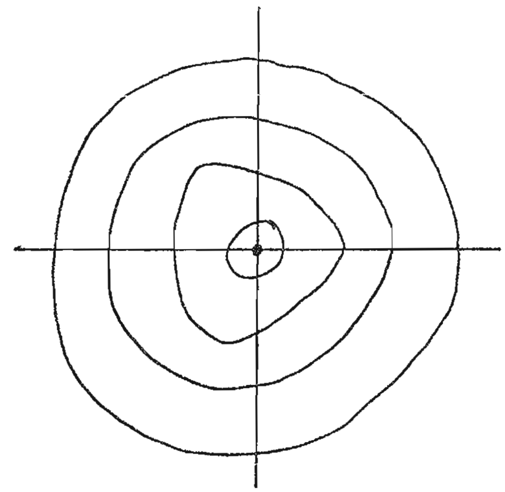

Fig. 2. Case a).

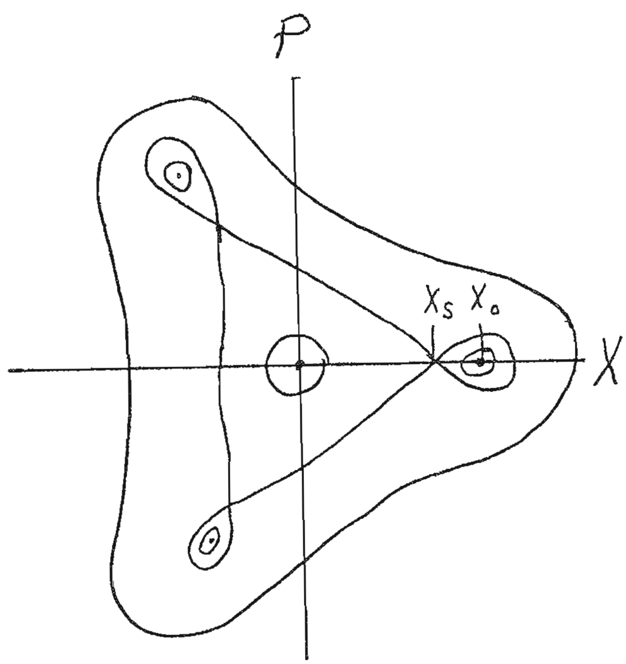

Fig. 3. Case b).

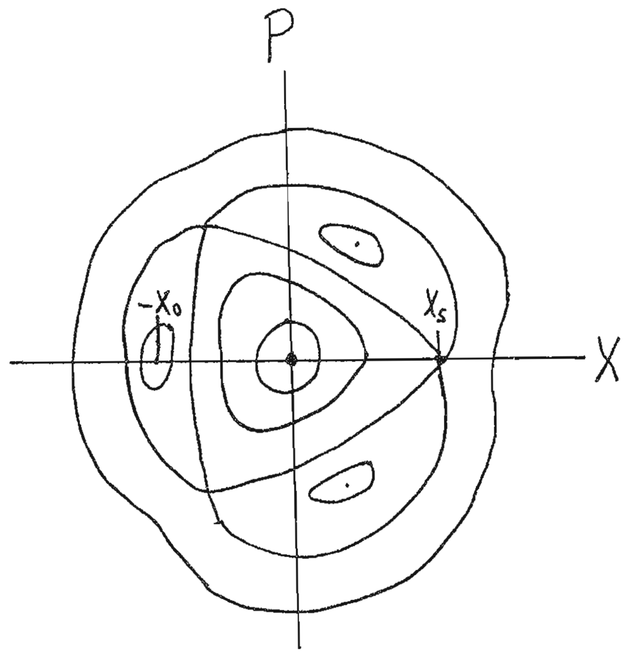

Fig. 4. Case c).

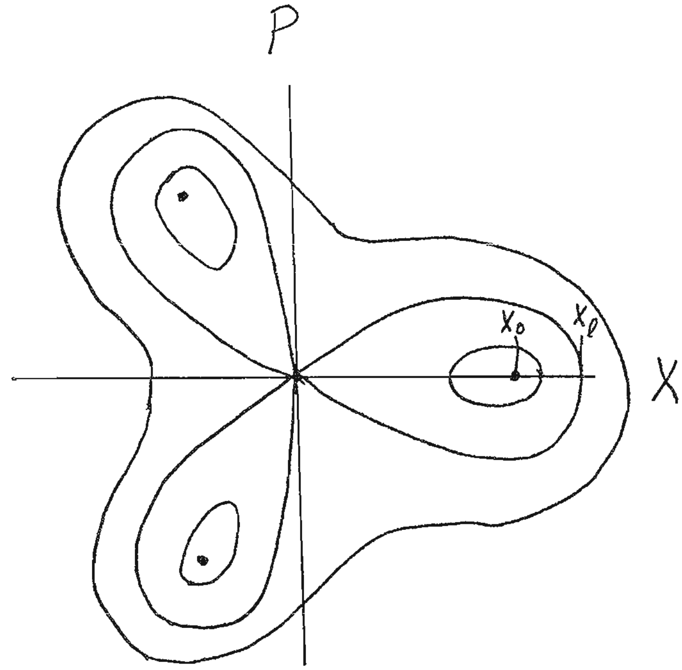

Fig. 5. Phase plane at resonance.