LS – 127
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Perturbation Analysis of the Octupole-induced Resonances in a Storage Ring

This note is a continuation from LS–126, in which we derived a formula for the lowest-order amplitude-dependent tune shift for octupole-induced resonances. Here, we will apply the canonical perturbation theory to the octupolar Hamiltonian and attempt to extend our analysis further in order to obtain much clearer insight on the octupole-induced resonances. We will derive the distortion functions, which measure the distortions of the particle oscillation phase and amplitude in phase space. Based upon these distortion functions, we will derive the higher-order amplitude-dependent tune shifts for octupoles.

Our starting Hamiltonian is Eq.(15) in LS–126. Since we already analyzed the ϕ-independent terms in LS–126, here we simply drop these terms and keep only those terms that depend on ϕ. Except for their contribution to the lowest-order amplitude-dependent tune shift, the ϕ-independent terms do not contribute to our analysis. The octupolar Hamiltonian V is then written as:

$$\begin{aligned} V(J,\phi;s) = \frac{B'''}{48B\rho}[&\beta_x^2 J_x^2(\cos 4\phi_x + 4\cos 2\phi_x) \\ &- 6\beta_x\beta_y J_x J_y\{\cos 2(\phi_x+\phi_y) + \cos 2(\phi_x-\phi_y) + 2\cos 2\phi_x \\ &+ 2\cos 2\phi_y\} + \beta_y^2 J_y^2(\cos 4\phi_y + 4\cos 2\phi_y)]. \end{aligned} \tag{1}$$

The above equation can be conveniently expressed by:

$$\begin{aligned} V(J,\phi;s) = &f_x(s)J_x^2(\cos 4\phi_x + 4\cos 2\phi_x) \\ &- 6f_{xy}(s)J_xJ_y(\cos 2\phi_+ + \cos 2\phi_- + 2\cos 2\phi_x + 2\cos 2\phi_y) \\ &+ f_y(s)J_y^2(\cos 4\phi_y + 4\cos 2\phi_y). \end{aligned} \tag{2}$$

where

$$
\begin{aligned}
f_x(s) &= \frac{1}{48}\frac{B'''(s)}{B\rho}\beta_x^2 \equiv \frac{1}{8}\underline{m}(s) \\
f_{xy}(s) &= \frac{1}{48}\frac{B'''(s)}{B\rho}\beta_x\beta_y \equiv \frac{1}{8}m(s) \\
f_y(s) &= \frac{1}{48}\frac{B'''(s)}{B\rho}\beta_y^2 \equiv \frac{1}{8}\bar{m}(s),
\end{aligned}
\tag{3}
$$

and $\underline{m}, m, \bar{m}$ are :

$$
\begin{aligned}
\underline{m} &\equiv \frac{B'''}{6B\rho}\beta_x^2 = \frac{B'''l}{6B\rho}\delta(s-s_k)\beta_x^2 \\
m &\equiv \frac{B'''}{6B\rho}\beta_x\beta_y = \frac{B'''l}{6B\rho}\delta(s-s_k)\beta_x\beta_y \\
\bar{m} &\equiv \frac{B'''}{6B\rho}\beta_y^2 = \frac{B'''l}{6B\rho}\delta(s-s_k)\beta_y^2.
\end{aligned}
\tag{4}
$$

In the above, s_k is the distance of the kth octupole from an arbitrary reference point in the ring.

The total Hamiltonian is then simply

$$
h = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} + V(J_x, J_y, \phi_x, \phi_y; s). \tag{5}
$$

For small V, we can treat the octupole term as a perturbation and make the Birkhoff–Moser transformation to J_{1x} and ϕ_{1x}. This transformation can be accomplished via the following generating function:

$$
F_1(\phi_x, \phi_y, J_{1x}, J_{1y}; s) = \phi_x J_{1x} + \phi_y J_{1y} + G(\phi_x, \phi_y, J_{1x}, J_{1y}; s) \tag{6}
$$

where G is the function that will be obtained in the following. The assumption of small V is valid as long as a particle is sufficiently far from the resonance.

If a particle is near the single resonance, then a canonical transformation exists that leaves only the dominant resonant term by transforming away all the non-resonant terms. Treatment of this case is not the subject of this note.

If G is small enough, the above transformation is close to the identity transformation:

$$\begin{aligned} \phi_{1z} &= \frac{\partial F_1}{\partial J_{1z}} = \phi_z + G_{J_{1z}} \\ J_z &= \frac{\partial F_1}{\partial \phi_z} = J_{1z} + G_{\phi_z} \quad , \end{aligned} \tag{7}$$

where $z = x, y$ and G_a is defined by:

$$G_a = \frac{\partial G}{\partial a} \quad . \tag{8}$$

The new Hamiltonian is then

$$\begin{aligned} h_2 = h + \frac{\partial F_1}{\partial s} &= \frac{J_{1x} + G_{\phi_x}}{\beta_x} + \frac{J_{1y} + G_{\phi_y}}{\beta_y} \\ &+ V(\phi_x, \phi_y, J_{1x} + G_{\phi_x}, J_{1y} + G_{\phi_y}; s) + G_s \quad . \end{aligned} \tag{9}$$

Following the assumption of small V, we expand V to first order in $\delta J_z = G_{\phi_z}$. When this is done, the new Hamiltonian has the form:

$$\begin{aligned} h_2 = &\frac{J_{1x} + G_{\phi_x}}{\beta_x} + \frac{J_{1y} + G_{\phi_y}}{\beta_y} + V(\phi_x, \phi_y, J_{1x}, J_{1y}) \\ &+ V_{J_{1x}} G_{\phi_x} + V_{J_{1y}} G_{\phi_y} + G_s. \end{aligned} \tag{10}$$

We note that the $V_{J_{1x}} G_{\phi_x}$ and $V_{J_{1y}} G_{\phi_y}$ terms are of second order. Therefore, if G satisfies the following equation,

$$\frac{G_{\phi_x}}{\beta_x} + \frac{G_{\phi_y}}{\beta_y} + G_s + V = 0, \tag{11}$$

then J_{1x} and J_{1y} are approximately constants of the motion to first order in δJ_x and δJ_y.

The new Hamiltonian then becomes

$$h_2 = \frac{J_{1x}}{\beta_x} + \frac{J_{1y}}{\beta_y} + V_{J_{1x}} G_{\phi_x} + V_{J_{1y}} G_{\phi_y}. \tag{12}$$

Let's consider Eq.(11). We'd like to find its periodic solution. Eq.(11) is, in more general form,

$$\nu(J_1) \cdot G_\phi + G_\theta + V(\phi, J_1; \theta) = 0 \quad , \tag{13}$$

where

$$\nu(J_1) = \frac{\partial h_2(J_1)}{\partial J_1} \quad .$$

Note that the independent variable has been changed from s to θ. In order for a periodic solution to exist, the following relation must hold:

$$\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi V(\phi, J; \theta) = 0 \quad . \tag{14}$$

Fourier-analyzing V and G, we have

$$\begin{aligned} V(\phi, J_1, \theta) &= \sum_k v_k(J_1, \theta) e^{ik\phi} \\ G(\phi, J_1, \theta) &= \sum_k g_k(J_1, \theta) e^{ik\phi} \\ G_\phi &= ik \sum_k g_k(J_1, \theta) e^{ik\phi} \\ G_\theta &= \sum_k \frac{\partial g_k(J_1, \theta)}{\partial \theta} e^{ik\phi} \quad . \end{aligned} \tag{15}$$

Therefore, Eq.(13) becomes

$$[ik\nu(J_1) + \frac{\partial}{\partial \theta}] g_k = -v_k \quad . \tag{16}$$

This equation can be solved by using the Green's function method. The solution

is given by

$$g_k = \frac{i}{2\sin\pi k\nu} \int_{\theta}^{\theta+2\pi} e^{ik[\phi+\nu(\theta'-\theta-\pi)]} v_k(J_1,\theta')d\theta'. \tag{17}$$

Using the above formalism and Fourier-transforming Eq.(2), we obtain

$$\begin{aligned} G = -\int_{s}^{s+C} [&\frac{f_x(s')J_x^2}{2}(\frac{\sin 4a_x}{\sin 4\pi\nu_x} + \frac{4\sin 2a_x}{\sin 2\pi\nu_x}) \\ &- 3f_{xy}(s')J_xJ_y(\frac{\sin 2a_+}{\sin 2\pi\nu_+} + \frac{\sin 2a_-}{\sin 2\pi\nu_-} + \frac{2\sin 2a_x}{\sin 2\pi\nu_x} \\ &+ \frac{2\sin 2a_y}{\sin 2\pi\nu_y}) + \frac{f_y(s')J_y^2}{2}(\frac{\sin 4a_y}{\sin 4\pi\nu_y} + \frac{4\sin 2a_y}{\sin 2\pi\nu_y})]ds', \end{aligned} \tag{18}$$

where

$$\begin{aligned} a_z &= \phi_z + \psi_z(s') - \psi_z(s) - \pi\nu_z \\ a_\pm &= \phi_\pm + \psi_\pm(s') - \psi_\pm(s) - \pi\nu_\pm \\ \phi_\pm &= \phi_x \pm \phi_y, \quad \psi_\pm = \psi_x \pm \psi_y, \quad \nu_\pm = \nu_x \pm \nu_y \end{aligned} \tag{19}$$

and

$$\begin{aligned} \psi_z(s) &= \int_0^s \frac{ds'}{\beta_z(s')} = \int_0^\theta \frac{Rd\theta'}{\beta_z(\theta')} \\ \nu_z &= \int_0^C \frac{ds'}{\beta_z(s')} = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} \frac{Rd\theta'}{\beta_z(\theta')}. \end{aligned} \tag{20}$$

where C is the circumference of the ring (*i.e.*, $C = 2\pi R$) and z denotes either x or y.

Equation (18) indicates that the octupole-induced resonances are, to first order,

$$\begin{aligned} &4\nu_x = p, \quad 2\nu_x = p, \quad 2\nu_x \pm 2\nu_y = p \\ &4\nu_y = p, \quad 2\nu_y = p \quad , \end{aligned} \tag{21}$$

where p is an integer.

Distortion Functions

Using Eqs.(7) and (18), we obtain the distortions in oscillation amplitude:

$$\delta J_x = \frac{\partial G}{\partial \phi_x} = -\int_s^{s+C} [2f_x(s')J_x^2(\frac{\cos 4a_x}{\sin 4\pi\nu_x} + \frac{2\cos 2a_x}{\sin 2\pi\nu_x}) \\ - 6f_{xy}(s')J_xJ_y(\frac{\cos 2a_+}{\sin 2\pi\nu_+} + \frac{\cos 2a_-}{\sin 2\pi\nu_-} + \frac{2\cos 2a_x}{\sin 2\pi\nu_x})]ds' \tag{22}$$

and

$$\delta J_y = \frac{\partial G}{\partial \phi_y} = -\int_s^{s+C} -[6f_{xy}(s')J_xJ_y(\frac{\cos 2a_+}{\sin 2\pi\nu_+} - \frac{\cos 2a_-}{\sin 2\pi\nu_-} + \frac{2\cos 2a_y}{\sin 2\pi\nu_y}) \\ + 2f_y(s')J_y^2(\frac{\cos 4a_y}{\sin 4\pi\nu_y} + \frac{2\cos 2a_y}{\sin 2\pi\nu_y})]ds'. \tag{23}$$

These equations can be written in more compact form:

$$\delta J_x = -2J_x^2(F_{4x} + 2F_{2x}) + 6J_xJ_y(F_{2+} + F_{2-} + 2\bar{F}_{2x}) \\ \delta J_y = 6J_xJ_y(F_{2+} - F_{2-} + 2\bar{F}_{2y}) - 2J_y^2(F_{4y} + 2F_{2y}) \tag{24}$$

where

$$F_{4z} = B_{4z}\cos 4\phi_z - A_{4z}\sin 4\phi_z \\ F_{2z} = B_{2z}\cos 2\phi_z - A_{2z}\sin 2\phi_z \\ \bar{F}_{2z} = \bar{B}_{2z}\cos 2\phi_z - \bar{A}_{2z}\sin 2\phi_z \\ F_{2\pm} = B_{2\pm}\cos 2\phi_\pm - A_{2\pm}\sin 2\phi_\pm \tag{25}$$

and

$$
\begin{aligned}
B_{4z} &= \int_s^{s+C} \frac{ds' f_z(s')}{\sin 4\pi\nu_z} \cos 4(\psi_z(s') - \psi_z(s) - \pi\nu_z) \\
A_{4z} &= \int_s^{s+C} \frac{ds' f_z(s')}{\sin 4\pi\nu_z} \sin 4(\psi_z(s') - \psi_z(s) - \pi\nu_z) \\
B_{2z} &= \int_s^{s+C} \frac{ds' f_z(s')}{\sin 2\pi\nu_z} \cos 2(\psi_z(s') - \psi_z(s) - \pi\nu_z) \\
A_{2z} &= \int_s^{s+C} \frac{ds' f_z(s')}{\sin 2\pi\nu_z} \sin 2(\psi_z(s') - \psi_z(s) - \pi\nu_z) \\
B_{2\pm} &= \int_s^{s+C} \frac{ds' f_{xy}(s')}{\sin 2\pi\nu_\pm} \cos 2(\psi_\pm(s') - \psi_\pm(s) - \pi\nu_\pm) \\
A_{2\pm} &= \int_s^{s+C} \frac{ds' f_{xy}(s')}{\sin 2\pi\nu_\pm} \sin 2(\psi_\pm(s') - \psi_\pm(s) - \pi\nu_\pm) \\
\bar{B}_{2z} &= \int_s^{s+C} \frac{ds' f_{xy}(s')}{\sin 2\pi\nu_z} \cos 2(\psi_z(s') - \psi_z(s) - \pi\nu_z) \\
\bar{A}_{2z} &= \int_s^{s+C} \frac{ds' f_{xy}(s')}{\sin 2\pi\nu_z} \sin 2(\psi_z(s') - \psi_z(s) - \pi\nu_z) \quad .
\end{aligned}
\tag{26}
$$

These functions are the distortion functions for normal octupoles. Distortion functions measure the distortion of the betatron oscillation amplitudes in phase-space.

Next, we'd like to express the above integrals in terms of the summations. For this purpose let us consider, for example, the B_{4z} term. The explicit expression for B_{4z} is given by :

$$B_{4z} = \frac{1}{48} \int_{s}^{s+C} \frac{\frac{B'''(s)}{B\rho} l \beta_z^2 \delta(s' - s_k)}{\sin 4\pi\nu_z} ds' \cos 4(\psi_z(s') - \psi_z(s_k) - \pi\nu_z).$$

For $s' < s_k$,

$$\psi_z(s_k) = \psi_z(s_k) + 2\pi\nu_z \quad .$$

Therefore, for $s' < s_k$,

$$B_{4z} = \frac{1}{48} \sum_k \frac{\frac{B'''}{B\rho}\beta_z^2}{\sin 4\pi\nu_z} \cos 4(\psi_z(s') - \psi_z(s_k) + \pi\nu_z) \quad .$$

On the other hand, for $s' > s_k$,

$$B_{4z} = \frac{1}{48} \sum_k \frac{\frac{B'''}{B\rho}\beta_z^2}{\sin 4\pi\nu_z} \cos 4(\psi_z(s') - \psi_z(s_k) - \pi\nu_z) \quad .$$

In general,

$$\begin{aligned}
B_{4x}(s) &= \frac{1}{8} \sum_k \frac{\underline{m}_k}{\sin 4\pi\nu_x} \cos 4(|\psi_x(s_k) - \psi_x(s)| - \pi\nu_x) \\
B_{4y}(s) &= \frac{1}{8} \sum_k \frac{\bar{m}_k}{\sin 4\pi\nu_y} \cos 4(|\psi_y(s_k) - \psi_y(s)| - \pi\nu_y) \\
B_{2x}(s) &= \frac{1}{8} \sum_k \frac{\underline{m}_k}{\sin 2\pi\nu_x} \cos 2(|\psi_x(s_k) - \psi_x(s)| - \pi\nu_x) \\
B_{2y}(s) &= \frac{1}{8} \sum_k \frac{\bar{m}_k}{\sin 2\pi\nu_y} \cos 2(|\psi_y(s_k) - \psi_y(s)| - \pi\nu_y) \\
B_{2\pm}(s) &= \frac{1}{8} \sum_k \frac{m_k}{\sin 2\pi\nu_\pm} \cos 2(|\psi_\pm(s_k) - \psi_\pm(s)| - \pi\nu_\pm) \\
\bar{B}_{2x}(s) &= \frac{1}{8} \sum_k \frac{m_k}{\sin 2\pi\nu_x} \cos 2(|\psi_x(s_k) - \psi_x(s)| - \pi\nu_x) \\
\bar{B}_{2y}(s) &= \frac{1}{8} \sum_k \frac{m_k}{\sin 2\pi\nu_y} \cos 2(|\psi_y(s_k) - \psi_y(s)| - \pi\nu_y).
\end{aligned} \tag{27}$$

In the above equations, $\underline{m}, m, \bar{m}$ are given by Eq.(4).

We now express the distortions of the amplitude in terms of the real coordinates. Rewriting Eq.(24) explicitly in terms of the distortion functions yields:

$$\begin{aligned}\delta J_x &= -2J_x^2(B_{4x}\cos 4\phi_x - A_{4x}\sin 4\phi_x + 2B_{2x}\cos 2\phi_x \\ &\quad - 2A_{2x}\sin 2\phi_x) + 6J_xJ_y(B_{2+}\cos 2\phi_+ - A_{2+}\sin 2\phi_+ + B_{2-}\cos 2\phi_- \\ &\quad - A_{2-}\sin 2\phi_- + 2\bar{B}_{2x}\cos 2\phi_x - 2\bar{A}_{2x}\sin 2\phi_x)\end{aligned} \tag{28}$$

and

$$\begin{aligned}\delta J_y &= 6J_xJ_y(B_{2+}\cos 2\phi_+ - A_{2+}\sin 2\phi_+ - B_{2-}\cos 2\phi_- \\ &\quad + A_{2-}\cos 2\phi_- + 2\bar{B}_{2y}\cos 2\phi_y - 2\bar{A}_{2y}\sin 2\phi_y) \\ &\quad - 2J_y^2(B_{4y}\cos 4\phi_y - A_{4y}\sin 4\phi_y + 2B_{2y}\cos 2\phi_y - 2A_{2y}\sin 2\phi_y).\end{aligned} \tag{29}$$

The above two equations can be expressed in terms of the particle oscillation amplitude, $A_z = \sqrt{2\beta_z J_z}$:

$$\delta A_z = \frac{\beta_z}{A_z}\delta J_z, \quad \sqrt{2J_z} = \frac{A_z}{\sqrt{\beta_z}} \quad . \tag{30}$$

Therefore, we obtain

$$\begin{aligned}\delta A_x &= \frac{\beta_x}{A_x}\delta J_x = -\frac{A_x^3}{2\beta_x}(B_{4x}\cos 4\phi_x - A_{4x}\sin 4\phi_x \\ &\quad + 2B_{2x}\cos 2\phi_x - 2A_{2x}\sin 2\phi_x) + \frac{3A_xA_y^2}{2\beta_y}(B_{2+}\cos 2\phi_+ \\ &\quad - A_{2+}\sin 2\phi_+ + B_{2-}\cos 2\phi_- - A_{2-}\sin 2\phi_- + 2\bar{B}_{2x}\cos 2\phi_x - 2\bar{A}_{2x}\sin 2\phi_x)\end{aligned} \tag{31}$$

and

$$\begin{aligned}\delta A_y &= \frac{\beta_y}{A_y}\delta J_y = \frac{3A_x^2A_y}{2\beta_x}(B_{2+}\cos 2\phi_+ - A_{2+}\sin 2\phi_+ \\ &\quad - B_{2-}\cos 2\phi_- + A_{2-}\sin 2\phi_- + 2\bar{B}_{2y}\cos 2\phi_y - 2\bar{A}_{2y}\sin 2\phi_y) \\ &\quad - \frac{A_y^3}{2\beta_y}(B_{4y}\cos 4\phi_y - A_{4y}\sin 4\phi_y + 2B_{2y}\cos 2\phi_y - 2A_{2y}\sin 2\phi_y)\end{aligned} \tag{32}$$

where β_x, β_y are the β-functions at the observation point.

Amplitude-dependent Tune Shift

Consider the new Hamiltonian given by Eq.(9). This can be written in the form:

$$h_2 = h_0(J_1) + \nu(J_1) \cdot G_\phi + G_s + V(\phi, J_1; s) + V_{J_1} \cdot G_\phi + \cdots. \tag{33}$$

In the above equation, the nonlinear term can be separated into a part that depends only on the new action variable and into another part that involves J_1, ϕ_1, and s but has an average value of zero. This oscillatory term is the object of the next canonical transformation. The term, which is a new action variable J_1, then leads to a change of frequencies with amplitude.

The new Hamiltonian can be written in the form,

$$\begin{aligned} h_2 &= h_0(J_1) + < V'(J_1) > + [V' - < V' >] \\ &\equiv h_{01}(J_1) + V_1(\phi_1, J_1, s) \end{aligned} \tag{34}$$

and the new shifted frequency is

$$\nu_1(J_1) = \frac{\partial h_{01}}{\partial J_1} = \nu(J_1) + \frac{\partial < V' >}{\partial J_1}. \tag{35}$$

Eq.(34) is explicitly given by

$$h_{01} = \frac{J_{1x}}{\beta_x} + \frac{J_{1y}}{\beta_y} + \frac{1}{C} \int_0^C (W_x + W_y) ds, \tag{36}$$

where

$$W_z = < G_{\phi_z} J_{1z} > \quad , z = x, y \quad .$$

From Eq.(2),

$$V_{J_{1x}} = \frac{\partial V}{\partial J_x} = 2f_x J_x(\cos 4\phi_x + 4\cos 2\phi_x) \\ - 6f_{xy}J_y(\cos 2\phi_+ + \cos 2\phi_- + 2\cos 2\phi_x + 2\cos 2\phi_y). \tag{37}$$

On the other hand, $G_{\phi_x} = (\partial G/\partial \phi_x) = \delta J_x$ has already been calculated and is given by Eq.(24). In what follows, we will omit all the A terms, simply because they do not contribute to the average. We then have

$$G_{\phi_x} \equiv -2J_x^2(B_{4x}\cos 4\phi_x + 2B_{2x}\cos 2\phi_x) \\ + 6J_xJ_y(B_{2+}\cos 2\phi_+ + B_{2-}\cos 2\phi_- + 2\bar{B}_{2x}\cos 2\phi_x). \tag{38}$$

Thus,

$$\begin{aligned} W_x =< G_{\phi_x}V_{J_{1x}} >= & \frac{1}{C}\int_0^C [-2J_x^2(B_{4x}\cos 4\phi_x + 2B_{2x}\cos 2\phi_x) \\ & + 6J_xJ_y(B_{2+}\cos 2\phi_+ + B_{2-}\cos 2\phi_- + 2\bar{B}_{2x}\cos 2\phi_x)] \\ & \times [2f_xJ_x(\cos 4\phi_x + 4\cos 2\phi_x) - 6f_{xy}J_y(\cos 2\phi_+ \\ & + \cos 2\phi_- + 2\cos 2\phi_x + 2\cos 2\phi_y]ds \\ = & \frac{1}{C}\int_0^C [-4f_xJ_x^3(B_{4x}\cos^2 4\phi_x + 8B_{2x}\cos^2 2\phi_x) \\ & + 12f_xJ_x^2J_y(8\bar{B}_{2x}\cos^2 2\phi_x) + 12f_{xy}J_x^2J_y(4B_{2x}\cos^2 2\phi_x) \\ & - 36f_{xy}J_xJ_y^2(B_{2+}\cos^2 2\phi_+ + B_{2-}\cos^2 2\phi_- \\ & + 4\bar{B}_{2x}\cos^2 2\phi_x)]ds. \end{aligned} \tag{39}$$

After taking the average, we obtain

$$W_x = -2f_xJ_x^3(B_{4x} + 8B_{2x}) + 48f_xJ_x^2J_y\bar{B}_{2x} \\ + 24f_{xy}J_x^2J_yB_{2x} - 18f_{xy}J_xJ_y^2(B_{2+} + B_{2-} + 4\bar{B}_{2x}). \tag{40}$$

We now consider $f_x\bar{B}_{2x}$ and $f_{xy}B_{2x}$.

$$
\begin{aligned}
f_x\bar{B}_{2x} =& \frac{1}{48}\frac{B'''(s)l\beta_x^2}{B\rho}\delta(s-s_k)\frac{1}{48}\int_s^{s+C}\frac{ds'\frac{B'''(s)l}{B\rho}}{\sin 2\pi\nu_x}\beta_x\beta_y\delta(s-s_k) \\
&\times\cos 2(\psi_x(s')-\psi_x(s)-\pi\nu_x) \\
=& \sum_k \frac{1}{48}(\frac{B'''l}{B\rho}\beta_x\beta_y)_k\delta(s-s_k)\frac{1}{48}\frac{1}{\sin 2\pi\nu_x}(\frac{\beta_x^2B'''l}{B\rho})_k \\
&\times\cos 2(\psi_x(s_k)-\psi_x(s)-\pi\nu_x).
\end{aligned}
\tag{41}
$$

This holds for W_x. Therefore, $f_x\bar{B}_{2x} = f_{xy}B_{2x}$. Thus,

$$
W_x = -2f_xJ_x^3(B_{4x}+8B_{2x})+72f_{xy}J_x^2J_yB_{2x}-18f_{xy}J_xJ_y^2(B_{2+}+B_{2-}+4\bar{B}_{2x}) \tag{42}
$$

We can also obtain W_y by the same procedure.

$$
\begin{aligned}
W_y =< G_{\phi_y}V_{J_{1y}} >=& \frac{1}{C}\int_0^C[6J_xJ_y(B_{2+}\cos 2\phi_+ - B_{2-}\cos 2\phi_-) \\
&+ 2\bar{B}_{2y}\cos 2\phi_y) - 2J_y^2(B_{4y}\cos 4\phi_y + 2B_{2y}\cos 2\phi_y)] \\
&\times[-6f_{xy}J_x(\cos 2\phi_+ + \cos 2\phi_-) + 2\cos 2\phi_x + 2\cos 2\phi_y) \\
&+ 2f_xf_y(\cos 4\phi_y + 4\cos 2\phi_y)]ds \\
=& \frac{1}{C}\int_0^C[-36f_{xy}J_x^2J_y(B_{2+}\cos^2 2\phi_+ \\
&- B_{2-}\cos^2 2\phi_- + 4\bar{B}_{2y}\cos^2 2\phi_y) \\
&+ 12f_{xy}J_xJ_y^2(4B_{2y}\cos^2 2\phi_y) + 12f_yJ_xJ_y^2(8\bar{B}_{2y}\cos^2 2\phi_y) \\
&- 4f_yJ_y^3(B_{4y}\cos^2 4\phi_y + 8B_{2y}\cos^2 2\phi_y)]ds. \\
=& -18f_{xy}J_x^2J_y(B_{2+} - B_{2-} + 4\bar{B}_{2y}) \\
&+ 24f_{xy}J_xJ_y^2B_{2y} + 48f_yJ_xJ_y^2\bar{B}_{2y} - 2f_yJ_y^3(B_{4y} + 8B_{2y}).
\end{aligned}
\tag{43}
$$

As in Eq.(41), in this case we find $f_{xy}B_{2y} = f_y\bar{B}_{2y}$. Therefore,

$$W_y = -18f_{xy}J_x^2J_y(B_{2+}-B_{2-}+4\bar{B}_{2y})+72f_{xy}J_xJ_y^2B_{2y}-2f_yJ_y^3(B_{4y}+8B_{2y}). \quad (44)$$

Finally, we have

$$\begin{aligned} W =& \frac{1}{C}\int_0^C (W_x + W_y)ds \\ =& \sum_k [-\frac{m}{4}J_x^3(B_{4x}+8B_{2x})_k - \frac{9}{4}mJ_x^2J_y(B_{2+}-B_{2-}-4B_{2x}+4\bar{B}_{2y})_k \\ & - \frac{9}{4}mJ_xJ_y^2(B_{2+}+B_{2-}+4\bar{B}_{2x}-4B_{2y})_k - \frac{1}{4}\bar{m}J_y^3(B_{4y}+8B_{2y})_k]. \end{aligned} \quad (45)$$

The amplitude-dependent tune shifts are, therefore, given by:

$$\begin{aligned} \Delta\nu_x = \frac{1}{2\pi}\frac{\partial W}{\partial J_x} =& -\frac{1}{2\pi}\sum_k [\frac{3m}{4}J_x^2(B_{4x}+8B_{2x})_k \\ & + \frac{9m}{2}J_xJ_y(B_{2+}-B_{2-}-4B_{2x}+4\bar{B}_{2y})_k \\ & + \frac{9m}{4}J_y^2(B_{2+}+B_{2-}+4\bar{B}_{2x}-4B_{2y})_k] \end{aligned} \quad (46)$$

and

$$\begin{aligned} \Delta\nu_y = \frac{1}{2\pi}\frac{\partial W}{\partial J_y} =& -\frac{1}{2\pi}\sum_k [\frac{9m}{4}J_x^2(B_{2+}-B_{2-} \\ & - 4B_{2x}+4\bar{B}_{2y})_k + \frac{9m}{2}J_xJ_y(B_{2+}+B_{2-} \\ & + 4\bar{B}_{2x}-4B_{2y})_k + \frac{3m}{4}J_y^2(B_{4y}+8B_{2y})_k]. \end{aligned} \quad (47)$$

Combining the above results with those given by Eq.(20) in LS-126, we finally

obtain the octupole-induced amplitude-dependent tune shifts :

$$
\begin{aligned}
2\pi\Delta\nu_x = & a^2\frac{3}{8}\sum \underline{m} - b^2\frac{3}{4}\sum m \\
& - \frac{3}{16}a^4\sum \underline{m}(B_{4x} + 8B_{2x}) \\
& - \frac{9}{8}a^2b^2\sum m(B_{2+} - B_{2-} - 4B_{2x} + 4\bar{B}_{2y}) \\
& - \frac{9}{16}b^4\sum m(B_{2+} + B_{2-} + 4\bar{B}_{2x} - 4B_{2y})
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
2\pi\Delta\nu_y = & -a^2\frac{3}{4}\sum m + b^2\frac{3}{8}\sum \bar{m} \\
& - \frac{9}{16}a^4\sum m(B_{2+} - B_{2-} - 4B_{2x} + 4\bar{B}_{2y}) \\
& - \frac{9}{8}a^2b^2\sum m(B_{2+} + B_{2-} + 4\bar{B}_{2x} - 4B_{2y}) \\
& - \frac{3}{16}b^4\sum \bar{m}(B_{4y} + 8B_{2y}).
\end{aligned}
\tag{49}
$$