

More Thoughts on the Aladdin Experiments

Experiment Set 2

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1. Eliminate octupole terms.

It turns out that the theorem of corresponding motions, mentioned in Ref. 1, is true only if we restrict the equations of motion to linear and sextupole terms. It is not necessary to work in a regime where the theorem holds, but it has two big advantages. It allows an easy check (see if the theorem holds) on whether there are important terms we are not keeping in the analysis. It also means we can survey the entire neighborhood of the resonance intersection by surveying a semicircle around the intersection. I will assume that we choose the intersection $\nu_x = 7 \frac{1}{6}$, $\nu_z = 7 \frac{1}{3}$ as suggested in Ref. 1, and that we maintain the validity of the theorem of corresponding motions.

Only the two third-integral resonances $3 \nu_z = 22$ and $2 \nu_x - \nu_z = 7$ may then enter the equations. We do not want any octupole terms. Since the frequency vs amplitude terms are DC terms of octupole order, we must adjust the main sextupoles to reduce the frequency shift vs amplitude so that frequency shifts are negligible out to the dynamic aperture, or at least so that they are dominated by the effects of the resonance terms.

2. Character of the Motion.

In Fig. 1, I show the relevant portion of the tune diagram, i.e., near the resonance intersection. The origin is taken at the intersection, with axes

$$\epsilon_x = \nu_x - 7 \frac{1}{6} \quad , \quad \epsilon_z = \nu_z - 7 \frac{1}{3} \quad (2.1)$$

The two resonance lines are shown, corresponding to the resonances

$$\begin{aligned}
 3\nu_z &= 22, \\
 2\nu_x - \nu_z &= 7.
 \end{aligned}
 \tag{2.2}$$

Also sketched is a semicircle around which we want to survey the character of the motion. The semicircle is presumed close enough to the intersection so the theorem of corresponding motions holds. The dashed lines are the bisectors of the angles between the resonance lines. The quadrant (d,e) within these bisectors is expected to be dominated (more or less) by the resonance $3\nu_z = 22$. The other quadrant (a,b,c) is expected to be dominated by the difference resonance. We will see that in the quadrant (b,c,d) where ϵ_x and ϵ_z have the same sign, the motion is stable at small amplitudes. In the other quadrant (a,e), where ϵ_x and ϵ_z have opposite signs, the constant of the motion turns out to be indefinite even at infinitesimal amplitudes, so we can say nothing definite about the stability.

We expect that near the resonance $3\nu_z = 22$, there will be a threshold instability amplitude z_s , depending on ϵ_z , above which the z-motion will be unstable. Whether this is true throughout the regions d and e, and what role the x-motion plays, is to be determined by tracking.

In regions b and c, we expect the difference resonance will have a threshold z-amplitude z_t above which small amplitude x-motion is unstable and exchanges energy with the z-motion. The constant of the motion will also give us an amplitude z_a above which the amplitudes are energetically allowed to grow indefinitely. If $z_t > z_a$, then if the experience described in Ref. 2 is a guide, if we start with a very small x-amplitude, and let the z-amplitude increase, we expect the motion to remain stable until $z = z_t$, beyond which the x-motion will grow because of the coupling resonance, and then become unstable because we are above the stability limit. If $z_t < z_a$, as it certainly will be very near the difference resonance, then for $z > z_t$, there will be x-z coupling, but the amplitudes cannot grow large until $z > z_a$. The experience described in Ref. 2 suggests that just above z_a , there will be a very narrow pass in phase space leading into the large amplitude region, so that the motion may appear to be stable for many revolutions before suddenly growing to a large amplitude. This is the place where tracking methods may fail; the

failure can be recognized by the fact that the apparent dynamic aperture becomes an erratic function of the initial conditions. As z increases above z_a , the pass opens up, and the number of turns to become unstable decreases rapidly, so that tracking a few turns discloses the instability.

In region a, instability may energetically occur at very small amplitudes, but the experience of Ref. 2 suggests that it probably does not, in fact, occur until the threshold z_t for the difference resonance is exceeded. These speculations need to be confirmed by tracking studies.

3. Analysis of the motion.

In the neighborhood of the resonance intersection, we can write a Hamiltonian which contains the relevant terms. It is convenient to write it in terms of the angle-action variables for the linear motion:

$$\begin{aligned}
 H = & \nu_x J_x + \nu_z J_z + A(2J_z)^{3/2} \sin(3\gamma_z - 22\theta) + B(2J_x)^{1/2} (2J_z) \sin(2\gamma_x - \gamma_z - 7\theta) \\
 & + \frac{1}{2} C J_x^2 + D J_x J_z + \frac{1}{2} E J_z^2 + F J_x J_z \sin(2\gamma_z + 2\gamma_x - 29\theta). \quad (3.1)
 \end{aligned}$$

The independent variable here is $\theta = 2\pi s/C$, where C is the circumference of the reference orbit. The argument of each sine should contain a constant phase, but they are omitted here, as they are irrelevant for our considerations. For each degree of freedom, we may also introduce a rectangular coordinate and momentum via the equations

$$\begin{aligned}
 Q &= (2J)^{1/2} \sin\gamma, \\
 P &= (2J)^{1/2} \cos\gamma. \quad (3.2)
 \end{aligned}$$

Since the transformation (3.2) is canonical, the Hamiltonian (3.1) may also be written in terms of the canonical variables Q_x, P_x, Q_z, P_z . Note that Q, P can be regarded as rectangular coordinates, and $(2J)^{1/2}, \gamma$ as polar coordinates in the phase space for each degree of freedom. I have included also tune-shifting terms and the octupole term which drives the fourth-order

resonance through the intersection. These terms will be dropped later. We will use the Hamiltonian (3.1) to study the character of the motion. In order to relate the thresholds we derive to actual experiments, it will be necessary to start from the real Hamiltonian for Aladdin, make the necessary transformations to angle-action variables, and thus identify the coefficients A and B in terms of Aladdin parameters, as well as to determine the scaling between the variables P, Q, and the experimental variables x, x', z, z'. I will not take the time to carry out this development here.

We can eliminate the θ -dependence by transforming to coordinates rotating in each phase space at the frequencies $\nu_x = 7 \frac{1}{6}$, $\nu_z = 7 \frac{1}{3}$ corresponding to the intersection. We use the generating function

$$S = J_1(\gamma_x - (7 \frac{1}{6})\theta) + J_2(\gamma_z - (7 \frac{1}{3})\theta), \quad (3.3)$$

which leads to the equations

$$J_x = \frac{\partial S}{\partial \gamma_x} = J_1 \quad J_z = \frac{\partial S}{\partial \gamma_z} = J_2,$$

$$\gamma_1 = \frac{\partial S}{\partial J_1} = \gamma_x - (7 \frac{1}{6})\theta, \quad \gamma_2 = \frac{\partial S}{\partial J_2} = \gamma_z - (7 \frac{1}{3})\theta. \quad (3.4)$$

The Hamiltonian becomes

$$\begin{aligned} H_1 &= H + \frac{\partial S}{\partial \theta} \\ &= \epsilon_x J_1 + \epsilon_z J_2 + A(2J_2)^{3/2} \sin 3\gamma_2 + B(2J_1) (2J_2)^{1/2} \sin(2\gamma_1 - \gamma_2) \\ &\quad + \frac{1}{2} C J_1^2 + D J_1 J_2 + \frac{1}{2} E J_2^2 + F J_1 J_2 \sin(2\gamma_2 + 2\gamma_1), \end{aligned} \quad (3.5)$$

where

$$\epsilon_x = \nu_x - 7 \frac{1}{6}, \quad \epsilon_z = \nu_z - 7 \frac{1}{3}. \quad (3.6)$$

The new Hamiltonian is independent of θ and is therefore a constant of the motion. It is easily verified that if we keep only the terms in the second line, then if ϵ_x and ϵ_z are each changed by a factor α , and if the J's are changed by a factor α^2 , and θ by a factor $1/\alpha$, the resulting equations of motion are the same as those given by H_1 for $\alpha = 1$. This is the theorem of corresponding motions. It would not hold if we keep the terms in the third line. A corresponding theorem would hold, with a different scaling of J, if we had only linear and octupole terms (third line).

We now drop the octupole terms:

$$H_2 = \epsilon_x J_1 + \epsilon_z J_2 + A(2J_2)^{3/2} \sin 3\gamma_2 + B(2J_1)^{1/2} (2J_2) \sin(2\gamma_1 - \gamma_2). \quad (3.7)$$

In rectangular coordinates:

$$H_2 = \frac{1}{2} \epsilon_x (P_1^2 + Q_1^2) + \frac{1}{2} \epsilon_z (P_2^2 + Q_2^2) - A Q_2^3 + 3A Q_2 P_2^2 + B(Q_1^2 Q_2 - P_1^2 Q_2 + 2Q_1 P_1 P_2). \quad (3.8)$$

If we choose a working point near the resonance $3 \nu_z = 22$, i.e., near the ϵ_z -axis ($\epsilon_z = 0$), then we may expect this resonant term will dominate, and we may neglect the difference resonance term, to get the approximate Hamiltonian

$$H_3 = \epsilon_x J_1 + \epsilon_z J_2 + A(2J_2)^{3/2} \sin(3\gamma_2). \quad (3.9)$$

J_1 is now also a constant of the motion, and the problem can be solved exactly by the energy method. Curves of constant H_3 ($-\epsilon_z J_1$) are plotted in the Q_2, P_2 phase plane in Fig. 2. (Yes, Virginia, there is a Santa Claus; Eq. (3.9))

factors on the separatrix into a product of three straight lines.) The motion is stable out to the separatrix, beyond which it grows to large amplitudes. The fixed points are given by

$$\begin{aligned} J_s &= \epsilon_z^2 / 18A^2 \\ Q_s &= \epsilon_z / 3A, \\ P_s &= |\sqrt{3} \epsilon_z / 6A|. \end{aligned} \quad (3.10)$$

The figure is drawn for the case when ϵ_z and A have the same sign. At the resonance $\epsilon_z = 0$, the motion is unstable even at infinitesimal amplitudes, and even though the resonance is driven by a nonlinear term and the linear equations are stable! Otherwise, for small Q_2 , P_2 , the energy is positive definite, and the motion is stable. The dynamic aperture is given by $|Q_s|$, scaled to the experimental variable z . In Fig. 3, Q_s is plotted around the semicircle of Fig. 1.

It is easy to solve the equations of motion for the case $\epsilon_z = 0$, for motion out along the separatrix. The result is

$$J_2 = J_0 (1 - 12 \sqrt{2} A J_0^{1/2} \theta)^{-2}. \quad (3.11)$$

For small initial J_0 , the growth is very slow at first, but eventually the amplitude grows very rapidly, going to infinity at a finite value of θ . The solution (3.11) is also approximately valid for growth along a separatrix in Fig. 3, far from the fixed point.

Note that the above discussion is step 1 of the first set of experiments suggested in Ref. 1, except for scaling to experimental parameters.

A similar discussion can be given for motion near the difference resonance, in which we neglect the third term in Eq. (3.7), and transform to an angle variable $\gamma_c = 2\gamma_1 - \gamma_2$. The result is that there is no coupling at

small amplitudes of the x motion (Q_1, P_1). The threshold for coupling is

$$J_t = \frac{(2\epsilon_x - \epsilon_z)^2}{8B^2} \quad (3.12)$$

$$Q_t = \left| \frac{2\epsilon_x - \epsilon_z}{2B} \right|$$

Above this threshold, the coupling is strong, so that a small initial x-amplitude will grow until all the energy is in the x-motion, following which the amplitudes will oscillate as the energy is exchanged between the two degrees of freedom. In Fig. 3, Q_t is plotted around the semicircle of Fig. 1.

We now consider the case when both resonance terms in Eq. (3.7) are important. We then have only the one constant H, given by Eq. (3.7) or (3.8). If ϵ_x and ϵ_z have opposite signs (quadrant (a,e) in Fig. 1), then even the quadratic terms, which lead to the linear terms in the equations of motion, are of indefinite sign. The constancy of H then gives no information about the stability of the motion. If ϵ_x and ϵ_z have the same sign, then the quadratic terms have a definite sign (positive definite for the quadrant (b,c,d) in Fig. 1), and the motion must be stable at small amplitudes.

In order to analyze the latter case, we divide H into a kinetic and a potential energy:

$$H_2 = T_2 + V_2, \quad (3.13)$$

where

$$T_2 = \frac{1}{2} (\epsilon_x - 2BQ_2)P_1^2 + \frac{1}{2} (\epsilon_z + 6AQ_2)P_2^2 + 2BQ_1P_1P_2, \quad (3.14)$$

$$V_2 = \frac{1}{2} \epsilon_x Q_1^2 + \frac{1}{2} \epsilon_z Q_2^2 - AQ_2^3 + BQ_1^2 Q_2. \quad (3.15)$$

It is convenient to scale the variables in the following way. We first make the following definitions:

$$\epsilon_x = \epsilon \cos \zeta, \quad \epsilon_z = \epsilon \sin \zeta, \quad (3.16)$$

$$A = K \cos \beta, \quad B = K \sin \beta. \quad (3.17)$$

We then introduce scaled variables as follows:

$$Q_1 = (\epsilon/K)q_1, \quad Q_2 = (\epsilon/K)q_2, \quad P_1 = (\epsilon/K)p_1, \quad P_2 = (\epsilon/K)p_2, \quad (3.18)$$

$$\theta = \tau/\epsilon. \quad (3.19)$$

The scaled Hamiltonian for the motion as a function of the scaled independent variable τ is

$$h_2 = t_2 + v_2, \quad (3.20)$$

with

$$t_2 = \frac{1}{2}(\cos \zeta - 2 \sin \beta q_2)p_1^2 + \frac{1}{2}(\sin \zeta + 6 \cos \beta q_2)p_2^2 + 2 \sin \beta q_1 p_1 p_2, \quad (3.21)$$

$$v_2 = \frac{1}{2} \cos \zeta q_1^2 + \frac{1}{2} \sin \zeta q_2^2 - \cos \beta q_2^3 + \sin \beta q_1^2 q_2. \quad (3.22)$$

The theorem of corresponding motions is now explicitly exhibited. The motion of the scaled variables depends only on the angle ζ (Fig. 1), and on the angle β (essentially on the ratio A/B). As noted above, we are concerned only with the case when ζ is in the first quadrant. Reversing the signs of both $\sin \beta$ and $\cos \beta$ is equivalent to reversing the signs of both axes, so we may restrict ourselves to the case when β is in the first or second quadrant.

As long as h_2 increases with amplitude, the motion will be absolutely stable. In the first (and third) quadrant in Fig. 1, for small enough q_1 , q_2 , the kinetic energy is positive definite, and increases with p_1 and p_2 . In this case, we can study the stability by studying the potential energy.

The potential v_2 has four fixed points, (equilibrium points of the motion), given by

$$q_1 = q_2 = 0, \quad (3.23)$$

$$q_1 = 0, \quad q_2 = \frac{\sin\zeta}{3\cos\beta}, \quad (3.24)$$

and

$$q_2 = \frac{\cos\zeta}{2\sin\beta}, \quad q_1 = \pm \frac{(2\sin\zeta\cos\zeta)^{1/2}}{2\sin\beta} \left[1 + \frac{3}{2} \cot\zeta\cot\beta\right]^{1/2}. \quad (3.25)$$

To determine the nature of the fixed points, we examine the discriminant

$$\left(\frac{\partial^2 v_2}{\partial q_1 \partial q_2}\right)^2 - 4 \frac{\partial^2 v_2}{\partial q_1^2} \frac{\partial^2 v_2}{\partial q_2^2} = -4[\sin\zeta\cos\zeta + 2\sin\zeta\sin\beta(1-3\cot\zeta\cot\beta)q_2 - 12\sin\beta\cos\beta q_2^2 - \sin^2\beta q_1^2]. \quad (3.26)$$

The fixed point at the origin is elliptic. If $\tan\beta > 0$, the other three are hyperbolic. (The sign of $\tan\beta$ is the same as the sign of $\cos\beta$, in the first two quadrants.) If $\tan\beta < -(3/2)\cot\zeta$, fixed point (3.24) is hyperbolic, and the two points (3.25) do not exist. If $-(3/2)\cot\zeta < \tan\beta < 0$, the point (3.24) is elliptic, and the other two (3.25) are hyperbolic.

At large values of q_1, q_2 the cubic terms in v_2 are dominant:

$$v_2 \rightarrow q_2 (\sin\beta q_1^2 - \cos\beta q_2^2), \quad (3.27)$$

except near the asymptotes

$$q_2 = 0 \text{ and } q_2 = \pm(\tan\beta)^{1/2} q_1. \quad (3.28)$$

In Fig. 4 contour plots of $v_2(q_1, q_2)$ are sketched for the three cases.

The kinetic energy (3.21) is positive definite provided that the discriminant

$$4\sin^2\beta q_1^2 + 12\sin\beta\cos\beta[q_2 - \frac{\cos\zeta}{4\sin\beta}(1 - \frac{1}{3}\tan\beta\tan\zeta)]^2 - \cos\zeta\sin\zeta - \frac{3}{4}\cot\beta\cos^2\zeta(1 - \frac{1}{3}\tan\beta\tan\zeta)^2 < 0. \quad (3.29)$$

This condition is satisfied inside the circle defined by the left member. The circle is sketched in Fig. 4. In Fig. 4, we show the circle as lying entirely outside the inner separatrix, but for some values of β and ζ , the circle may intersect the inner separatrix. If the circle (3.28) lies outside the inner separatrix, then the separatrix represents the limiting value of h_2 inside which the motion must be stable. If the circle crosses inside the separatrix, then the limiting value of h_2 is given by the value of v_2 for the inner contour which is tangent to the circle.

In Fig. 3, I have sketched the value of $Q_2 = q_2/K$, corresponding to the fixed point on the inner separatrix in Fig. 4. To be precise, we should take Q_2 at the point where the inner separatrix crosses the q_2 -axis, or, if the circle (3.28) crosses the separatrix, then at the point where the tangent v_2 contour crosses the q_2 -axis; that would require more calculation than I am prepared to do at this point. In the quadrant (b,c,d), this represents a lower limit on the dynamical aperture. If the amplitude exceeds this limit,

then it is "energetically" possible for the amplitudes to grow large. Unless the orbit is kept away from the pass over the fixed point, for example due to lack of coupling, we expect the motion to be unstable. However, if the amplitude only slightly exceeds the limit, then the opening through the pass in the constant H_2 surface will be very small. In that case, unless the orbit is aimed at the fixed point, it may take a long time for the orbit to find it as it wanders over the constant H_2 surface. See the previous section for further discussion of this point.

4. Conclusion.

The above discussion is sufficient to show that experiments in the neighborhood of the intersection of the resonances (2.2) should exhibit a number of interesting behaviors. It remains to relate the variables Q, P to the experimental variables in Aladdin, and to determine the actual values of the coefficients A, B . In the interest of getting this report out as early as possible, I have not double-checked my calculations. If we decide to carry out the proposed experiments, the above calculations should be completed and carefully rechecked.

References.

1. K. Symon, University of Wisconsin-Madison, unpublished information (August, 1987).
2. H. K. Meier and K. R. Symon, "Analytical and Computational Studies on the Interaction of a Sum and a Difference Resonance," Proc. Int. Conf. on High-Energy Accelerators and Instrumentation, CERN 1959, pp. 253-262.

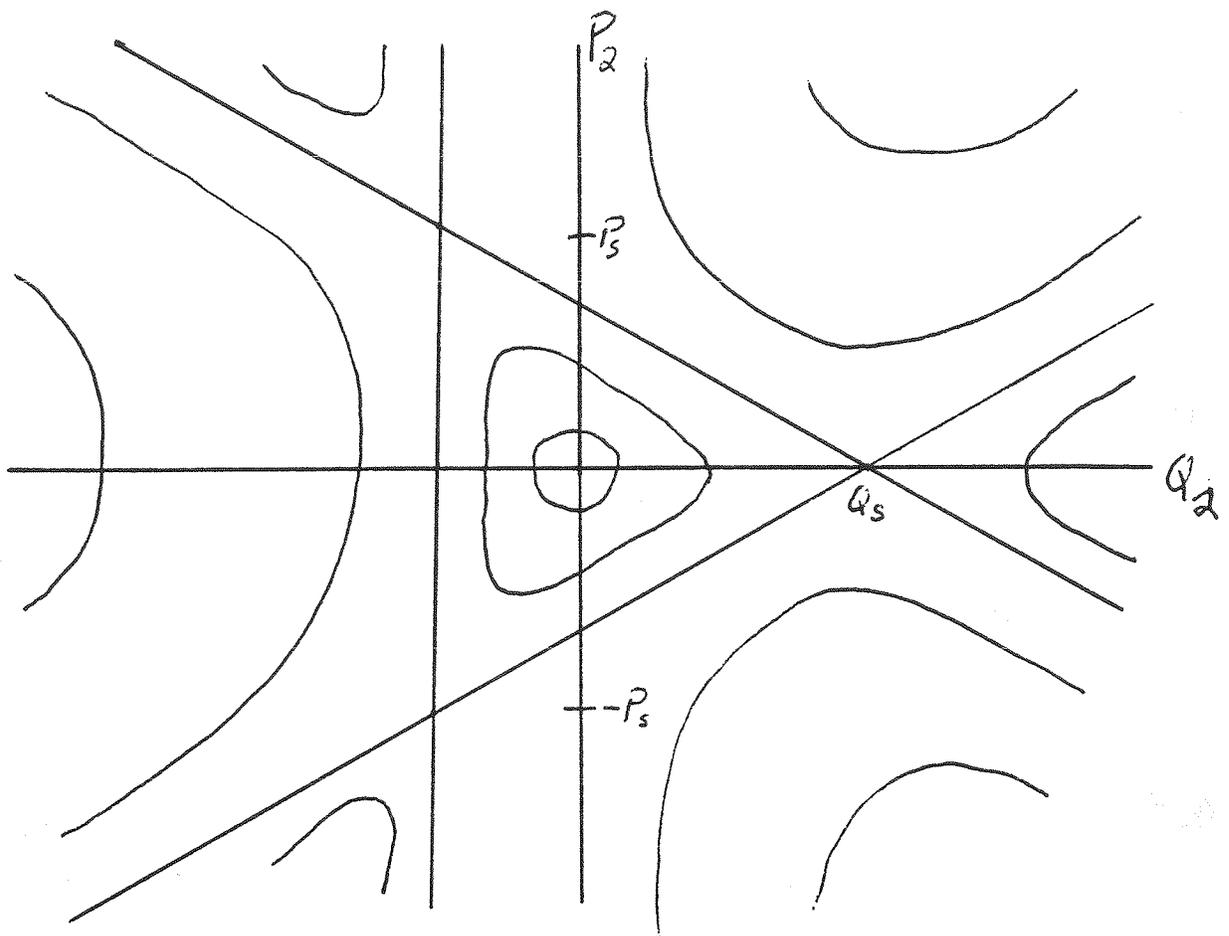


Fig. 2. Q_2, P_2 -phase plane near resonance $\nu_2 = 7\frac{1}{3}$.

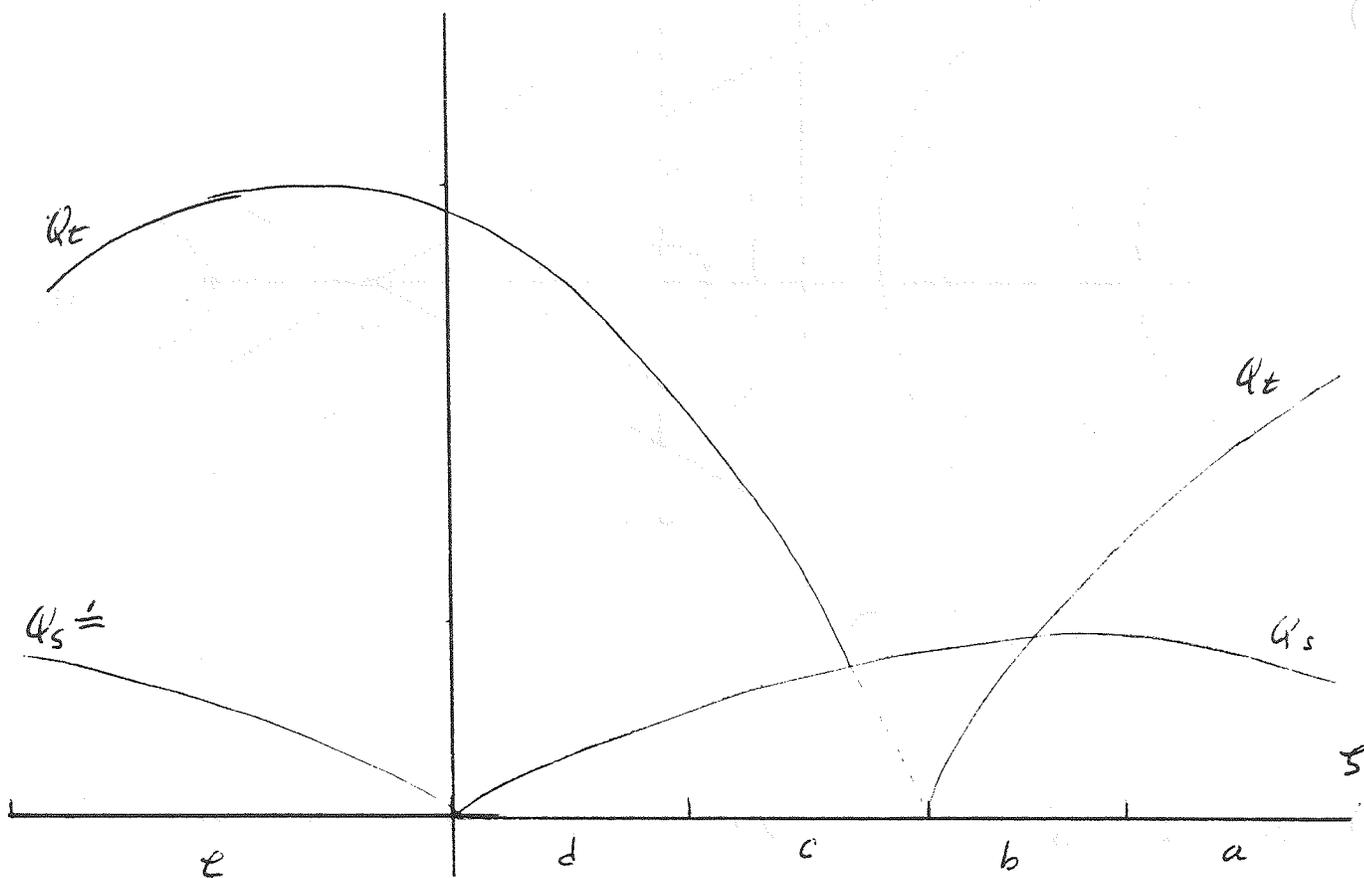
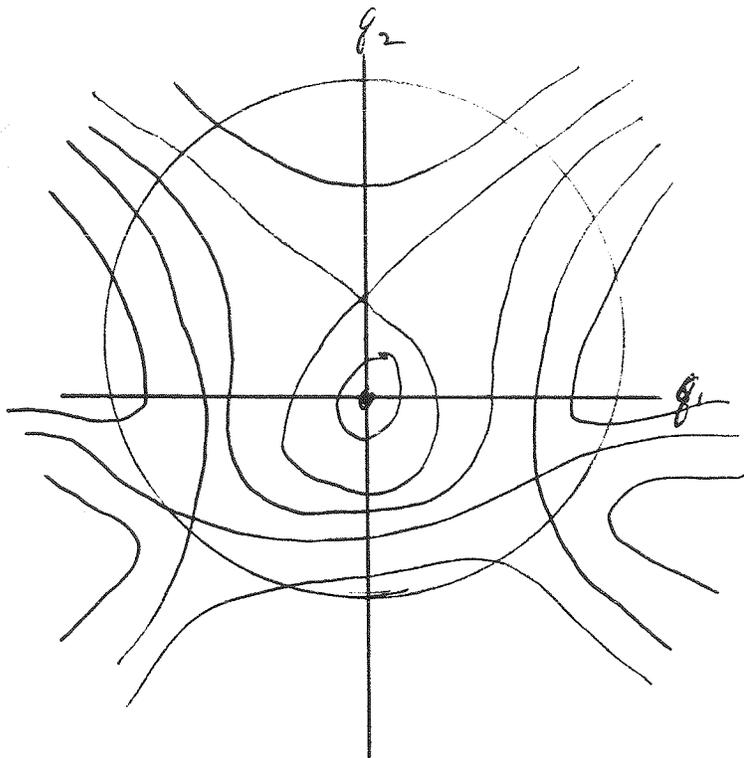


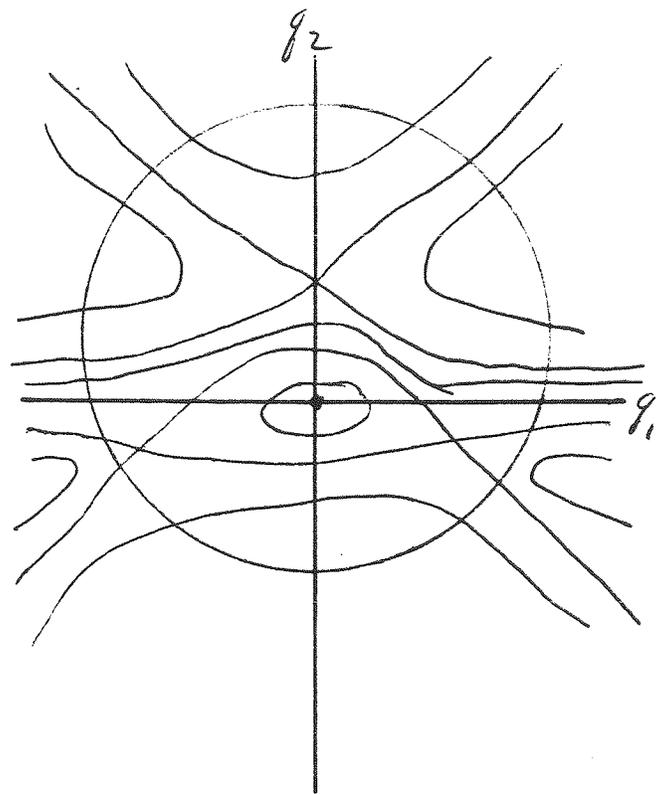
Fig. 3. Stability Thresholds.

$Q_x > Q_s$: $3\nu_2 = 22$ instability, and "energetically" unstable

$Q_x > Q_e$: $2\nu_x - \nu_2 = 7$ coupling resonance

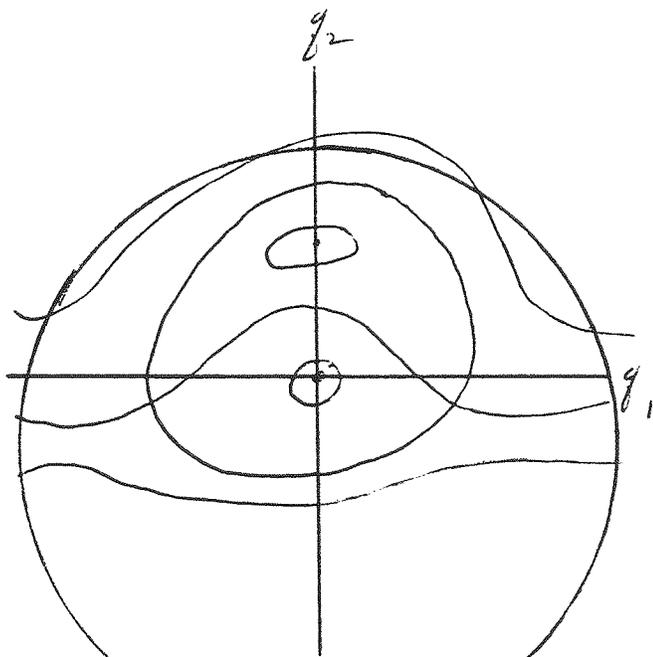


$$K < 1, \beta > 0$$

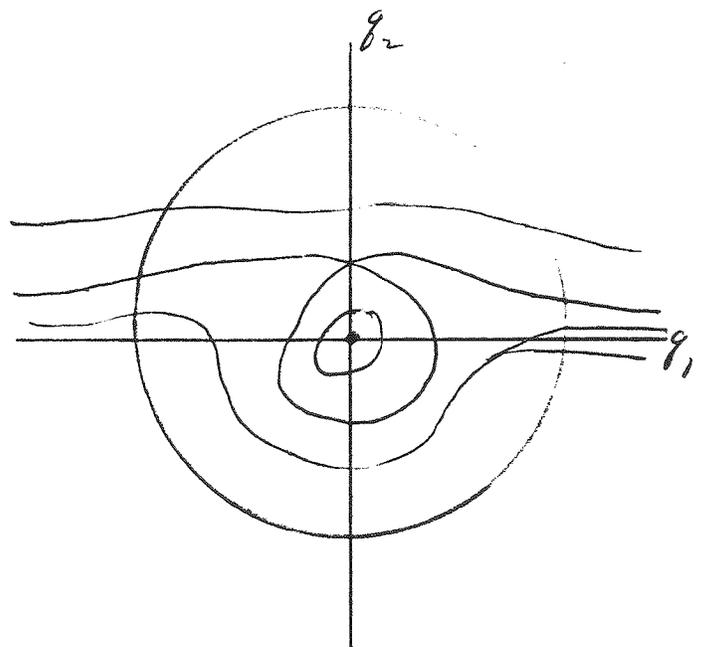


$$K > 1, \beta > 0$$

$$K = \frac{27}{4} \cot^2 \xi \cot^2 \beta [3 + 2 \tan \xi + [4 + 3 \tan \xi) \cot \xi \cot \beta]$$



$$\tan \beta < -\frac{3}{2} \cot \xi$$



$$-\frac{3}{2} \cot \xi < \tan \beta < 0$$

Fig. 4. Contour plots of $V_2(g_1, g_2)$.